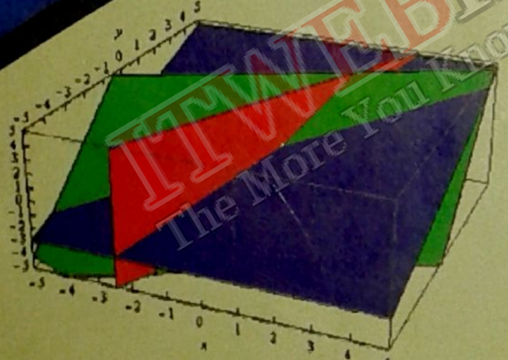
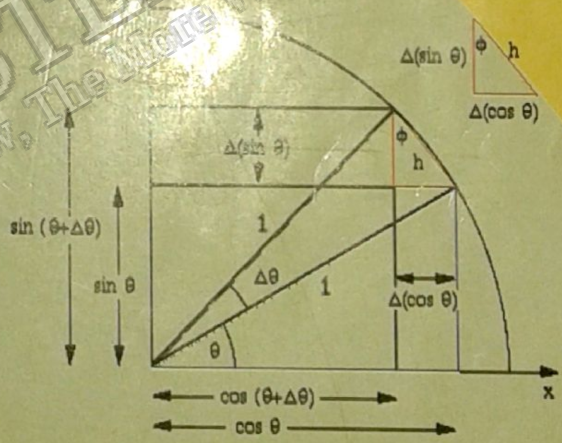
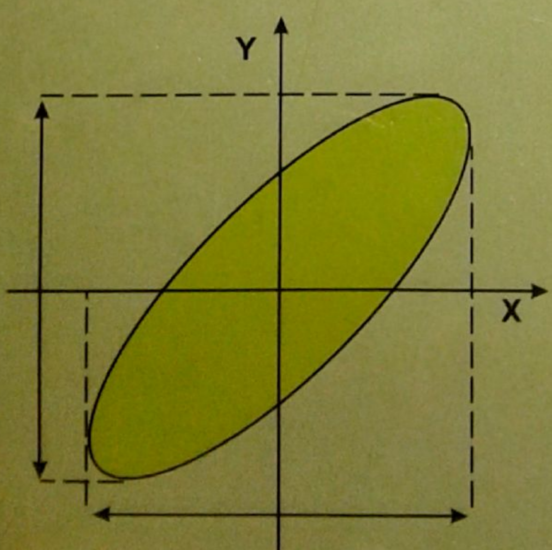
FARKALEET SERIES

Applied Calculus







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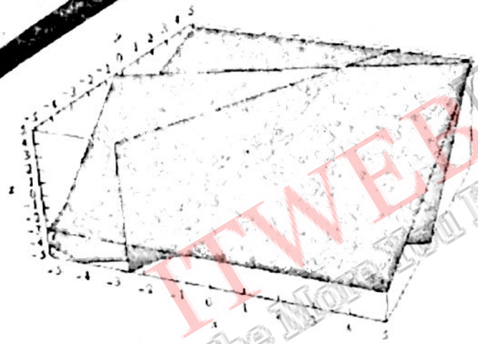

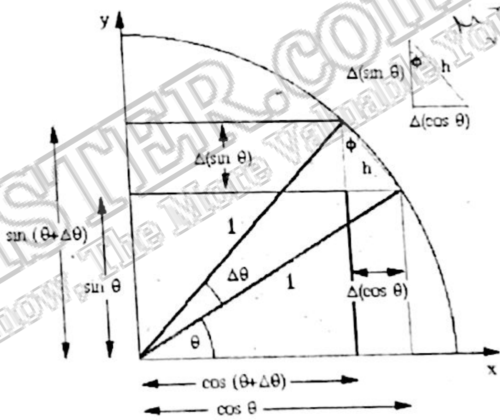

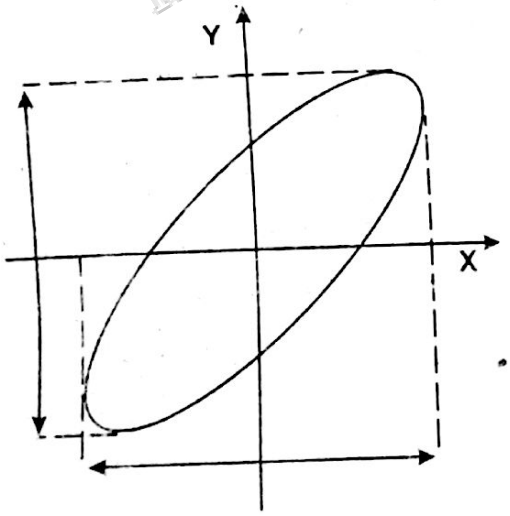

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PREFACE

About twenty years ago I wrote a series of books on different subjects of Engineering Mathematics under the name "FARKALEET SERIES". FARKELEET means MUHAMMAD (S.A.W.S). This name of our Prophet (S.A.W.S) is mentioned in the BIBLE.

The series became very much popular and liked by my students at Mehran. It gained popularity in students of other universities and colleges of Sindh as well. Many teachers of different universities and colleges also appreciated such efforts of mine. The reason is that each topic was clearly explained and that every student as well as teacher enjoyed reading the contents without any difficulty.

Afterwards I along with my colleagues put efforts as well as our teaching experience together and published different books on mathematics. Meanwhile, my students were insisting to rewrite this series once again as they wanted to be at ease while reading material on engineering mathematics.

I decided to write this series once again after my retirement. Now by the grace of Almighty ALLAH the most gracious and most merciful I have retired from my services and have started rewriting this series once again.

Now I myself along with my colleagues, Prof. Ashfaque Nabi Pathan, Chairman, BSRS, MUET, Jamshoro, Mr. Asif Ali Shaikh and Ms Sania Qureshi of BSRS, MUET, Jamshoro and Mr. Wajid Ali Shaikh of BSRS, QUEST, Nawabshah, Mr. Saeed Ahmed Rajput, Lecturer, BSRS, QUCEST, Larkana put the efforts to bring out this book on "APPLIED CALCULUS". You will find the experience of our whole life in this book. Each topic is explained in well manner and in detail as well.

We are indebted to our dearest friend and colleagues Prof. Khadim Hussain Bhutto, Chairman of Department of Mathematics and BSRS, QUEST, Nawabshah and Dr. Abdul Waseem Shaikh, Assistant Professor of IMCS, University of Sindh, Jamshoro, and Mr. Abdul Ghafoor Shaikh, Assistant Professor, BSRS, QUEST, Nawabshah who have always remained cooperative and well wisher of ours. With their inspiration it became possible for us to rewrite once again this series after many years.

We are thankful to all our colleagues of BSRS, MUET who encouraged and gave valuable suggestions which had made this book even more effective.

We shall be looking forward to hear from the readers any critic that will improve the standard of the book.

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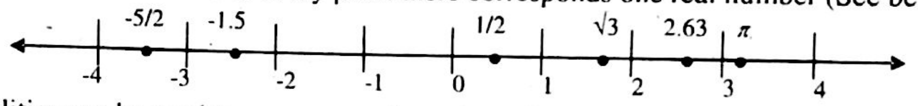
CONTENTS

CHAPTER ONE

FUNCTIONS AND THEIR APPLICATIONS

1.1 INTRIDUCTION TO REAL NUMBERS

The study of elementary calculus requires knowledge of the real number system. Real numbers can be considered as points on a line. For every real number there corresponds one point on this line and to every point there corresponds one real number (See below).



Inequalities can be used to compare real numbers. The symbols used are > (greater than), < (less than), ≥ (greater than or equal to), and ≤ (less than or equal to). For example, $x > 3$, $y < 4$, $x \geq 5$, $y \leq -2$, etc; are all inequalities.

In some applications, it is useful to combine two inequalities in order to express an **interval**. A **subset** of the real line is called an **interval** if it contains at least two different real numbers and other real numbers lying between them. For example, $2 < x < 5$ combines the inequalities $2 < x$ and $x < 5$ and represents all real numbers between 2 and 5. This is called open interval and we use the notation (2, 5). The open interval excludes the end points. Generally, this is expressed as:

$$(a, b) = \{x \mid a < x < b\}$$

The inequality $2 \leq x \leq 5$ represents a closed interval, one in which the end points are included. This interval is usually denoted by [2, 5]. Generally, this is expressed as:

$$[a, b] = \{x \mid a \leq x \leq b\}$$

The parentheses are used to indicate that an end point is not included. The square brackets are used to indicate that end–points are included. Intervals such as (2, 5] and [2, 5) are called half–open or half–closed intervals. The symbol ∞ (infinity) is used to specify that the interval extends infinitely far to the right. Similarly, - ∞ (minus infinity) is used to specify that an interval extends infinitely far to the left. Because ∞ does not represent a number, it is never included in the interval, and thus, a parenthesis is always used with this symbol.

Absolute Value

The absolute value of a real number x, denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

For example, $|3| = 3$, $|0| = 0$, $|-5| = 5$.

REMARK: (i) $|x| \geq 0$ for every real number x. **(ii)** If $|x| = 0$ then $x = 0$.

There are two other ways to define modulus of a real number x. They are:

(i) $|x| = \sqrt{x^2}$ and (ii) $|x| = \max(-x, x)$. For example,

(i) $|-4| = \sqrt{(-4)^2} = \sqrt{16} = 4$ and (ii) $|-4| = \max(-(-4), -4) = \max(+4, -4) = 4$.

Geometrically, $|x|$ represents the distance from the origin to x on the real line. More generally, $|x - y|$ = the distance between x and y.

Inequalities and Their Solutions

In this section we shall discuss how to solve an inequality problem. To solve inequalities, we first convert the inequality into an equation. This equation is known as "associated equation". Solutions of the associated equation are called "boundary numbers" for this inequality. If a rational expression occurs in equality, then the numbers where the denominator vanishes are not points in the domain of the rational expression. These numbers are called "free numbers". These numbers are not the part of the solution.

After finding the boundary numbers, locate them on the real line. The real line will be divided into a number of distinct "regions" each of which belongs to the solution set in its "entirety". Finally, we check the inequality by taking a point selected from each region. Union of all such regions constitutes a solution set of the inequality.

Example 01: Solve (i) $3x + 5 < x - 9$ (ii) $|5x + 6| \geq 5$ (iii) $x^2 - 5x + 6 < 0$ (iv) $x^2 - 2x + 2 > 0$ (v) $[(x^2 - 2)/(1 - 2x)] > 1$

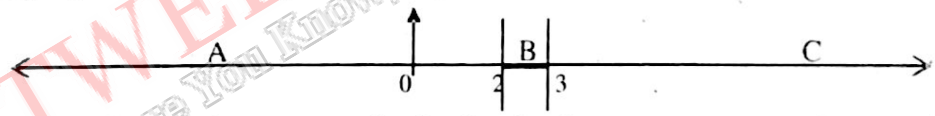
Solution: (i) $3x + 5 < x - 9$: Since $3x + 5 < x - 9 \Rightarrow 3x - x < -9 - 5 \Rightarrow 2x < -14$ $\Rightarrow x < -7$. Thus solution set is: $\{x \mid < -7\} = (-\infty, -7)$.

(ii) $|5x + 6| \geq 5$: By definition, if $|5x + 6| \geq 5$

$\Rightarrow$	$(5x + 6) \geq 5$	or	$-(5x + 6) \geq 5$	[See the technique]
$\Rightarrow$	$5x \geq 5 - 6$	or	$-5x - 6 \geq 5$	
$\Rightarrow$	$5x \geq -1$	or	$-5x \geq 5 + 6$	
$\Rightarrow$	$x \geq -1/5$	or	$x \leq -11/5$	

Thus the solution set is $(-\infty, -11/5] \cup [-1/5, \infty)$. This solution set is graphically shown below. The thick line shows the solution set of the given inequality.

(iii) $x^2 - 5x + 6 < 0$: The associated equation is $x^2 - 5x + 6 = 0 \Rightarrow x = 2$ and $x = 3$. These are the boundary numbers for the given inequality. The real line is divided into the following regions with the help of boundary numbers as shown below.



Region A:	Test $x = 0$	:	$0 - 0 + 6 = 6 < 0$	Result: False
Region B:	Test $x = 2.5$	:	$(2.5)^2 - 5(2.5) + 6 = -0.25 < 0$	Result: True
Region C:	Test $x = 4$	:	$4^2 - 5(4) + 6 = 2 < 6$	Result: False

Thus only the region B form the solution set. Therefore the solution set is: $S = (2, 3)$

It may be noted that if x is replaced by 2 and 3, the inequality will be not true. Hence the open interval (2, 3) is the only solution which is shown here in the form of thick line segment.

(iv) $x^2 - 2x + 2 > 0$: The associated equation is $x^2 - 2x + 2 = 0 \Rightarrow x = 1 \pm i$

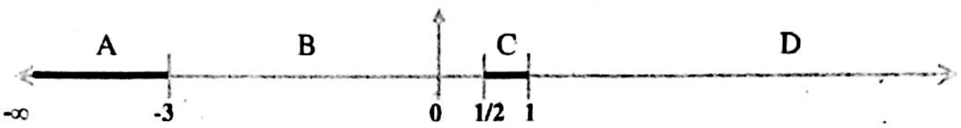
These are complex numbers which can not be represented on the real line. Thus there are no boundary numbers. We have therefore one region, that is, the entire real line as the given inequality is true for every real value of x.

(v) $[(x^2 - 2)/(1 - 2x)] > 1$: The associated equation is: $\left(x^2 - 2\right)/\left(1 - 2x\right) = 1$

$\Rightarrow x^2 - 2 = 1 - 2x \quad \Rightarrow x^2 + 2x - 3 = 0$. This gives: $x = -3, 1$

These are the boundary numbers for the given inequality. If the denominator of this inequality is zero $(1 - 2x = 0)$ then $x = 1/2$. This is the free boundary number for given inequality hence it can not be in the solution set.

The real line is divided by boundary numbers and free boundary numbers are shown in the following figure.



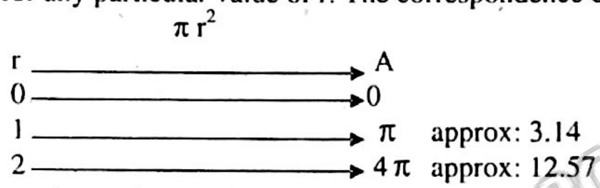
Region A: Test x = -4 : (16-2)/(1+8) > 1 Result: True
Region B: Test x = -1 : (1-2)/(1+2) > 1 Result: False
Region C: Test x = 3/4 : (9/16 -2)/(1-3/2)> 1 Result: True
Region D: Test x = 2 : (4-2)/(1-4) > 1 Result: False

The solution set is therefore: $S = (-\infty, -3) \cup (1/2, 1)$

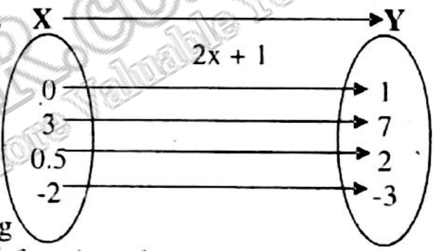
Graphically, the solution set is shown by thick line segments in the above figure.

1.2 INTRODUCTION TO FUNCTIONS

Recall the formula $A = \pi r^2$ which states that area A of the region within a circle is equal to π times the square of the radius r. The equation $A = \pi r^2$ relates two variables A and r such that for every non-negative value of r, there is a unique value of A. It indicates how to compute A for any particular value of r. The correspondence can be viewed as:



We observe that when r changes then A also changes. Moreover, for one value of r there is exactly one value of A. Such relation between r and A is defined as function.



An equation $y = 2x + 1$ also defines a function; because for any real x; there is a unique value of variable y. Some values of x and the corresponding values of y are shown in the figure. We say that y is a function of x.

Definition: A **function** is a rule of correspondence by which each element of set X is assigned to exactly one element of the other set Y.

In $A = \pi r^2$ the set of all possible input values for the radius is called domain of function. The set of all output values of area is called the range of function. Since circles cannot have negative radii or areas, the domain and range of this function is an interval $[0, \infty)$ consisting of all non–negative real numbers.

A Swiss mathematician, **Leonard Euler** (1707–1783) invented a symbolic way to say, "y is a function of x" by writing $y = f(x)$.

In this notation, the symbol f represents the function. The letter x, called the independent variable, represents an input value from the domain of f, and y the dependent variable, represents the corresponding output value in the range of f. In the set notation, function is written as:

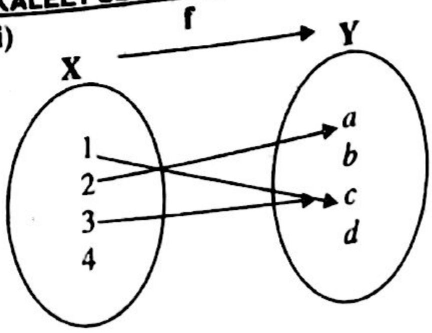
$$\{(x, f(x)) : x \in X, f(x) \in Y\}.$$

Symbolically, it is indicated by $f: X \to Y$ and is read as "f is a function from the set X to set Y".

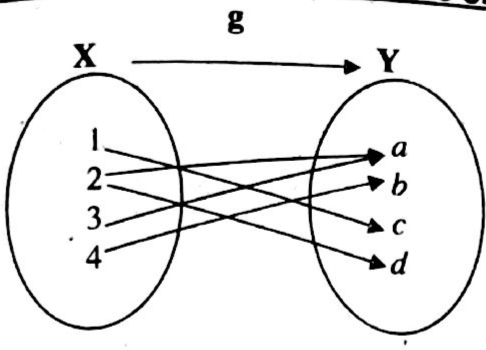
REMARK: **(i)** In calculus we deal with functions from R to R. **(ii)** Functions defined from R to R are called real-valued functions. **(iii)** Each element y of the set Y is called an image and each element x of the set X is called co-image of the function f.

Example 02: Which of the following diagrams describe a function from the set X = {1, 2, 3, 4} into Y{a, b, c, d}.

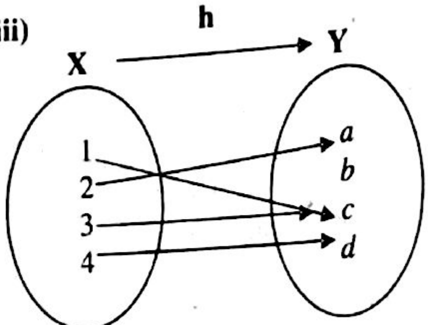
(i)



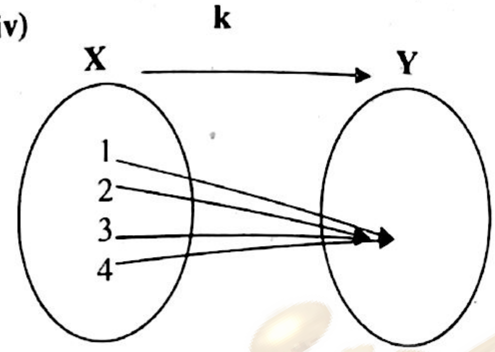
(ii)



(iii)



(iv)



Solution: (i) Relation f is not a function from X into Y as element 4 of set X is not assigned any of element of set Y.
(ii) Relation g is not a function from X into Y as element 2 of set X is assigned two different elements 'a' and 'd' of set Y and thus image of 2 is not unique.
(iii) Relation h is a function because every element of X has a unique image in Y.
(iv) Relation k is a function because every element of X has a unique image in Y.

Domain, Co-domain and Range of Real Valued Functions

Consider a real valued function $y = f(x)$ from set X to set Y. The set X is called **domain** and set Y is called **co-domain** of the function f. The set of those elements of Y which form the images of the elements of the set X is called the **range** of a function f. Refer to example 2 above we see that X = {1, 2, 3,4} is a domain and Y = (a, b, c, d} is a co-domain of functions h and k whereas, range of function h is the set {a, c, d} and range of function k is the set {c}.

Let $f(x) = 2x + 1$ be a function from R to R. Here x can be any real number and, when this number is used for x there will be a corresponding real number f(x) or y. The domain and range of this function f therefore, include all real numbers. The co-domain of this function is also a set of all real numbers.

Similarly let $y = f(x) = x^2$ be a function R to R. Here x can be any real number and, when this real number is used for x there will be a corresponding real number f(x) or y. The domain of this function is the set of all real numbers. But as we know that the square of a real number is always non–negative, therefore the range of this function will be the set of non–negative real numbers but the co-domain of this function is the set R.

Now consider a function $f(x) = x(2x + 3)$ where x represents the length of one side of a rectangle and (2x + 3) represents the length of another side of the rectangle. Here x can not be negative; therefore, the domain and range of this function are the set of non–negative real numbers.

REMARK: (i) The domain and range of a function f are usually denoted by D_f and R_f respectively. **(ii)** In addition to the nature of applications, there are other concerns that can restrict the domain of a function. These concerns are:

1. Division by zero

Any value of x that creates division by zero cannot be in the domain of a function. There would be no f(x) corresponding to such an x.

2. Square root of negative numbers

Any value of x creating the square root of a negative number can't be in the domain of a function. There would be no real f(x) corresponding to such real x.

Example 03: Find the domain of each of the following functions:

(a) $f(x)=\dfrac{1}{x-4}$ (b) $f(x)=\dfrac{x}{x^2+x}$ (c) $f(x)=\sqrt{x-1}$ (d) $f(x)=\sqrt{3-5x}$

Solution: (a) If we put x = 4, the denominator becomes zero. Since division by zero is not defined, no f value is produced if 4 is used for x. Thus, 4 is not in the domain of f. This means that the domain of f is the set of all real numbers except 4. Thus for this function $D_f = R - \{4\}$.

(b) Here the domain of f is the set of all real numbers except 0 and –1. We can write this simply as $x \neq 0, -1$. We may also say that $D_f = \mathbf{R} - \{0, -1\}$.

(c) If x - 1 is negative, the result is the square root of a negative number. Since the square root of a negative number is not a real number no f value will be produced in such case. This means that the domain is all x values for which $x - 1 \geq 0$ or $x \geq 1$. Thus, the domain of f is $x \geq 1$ or the interval $[1, \infty)$

(d) The domain of f is $3 - 5x \geq 0$. This means $3 \geq 5x$ Or $3/5 \geq x$ or $x \leq 3/5$. In the interval notation, we may write this as $D_f = (\infty, 3/5]$.

Example 04: Determine the domain of each of the following function:

(a) $f(x)=\sqrt{3+x}+\sqrt{7-x}$ (b) $f(x)=\sqrt{(x-4)/(x+1)}$

Solution: (a) Here f(x) is defined for:

$$3+x\geq 0 \text{ and } 7-x\geq 0 \Rightarrow x\geq -3 \text{ and } x\leq 7 \Rightarrow -3\leq x\leq 7 \text{ or } [-3,7].$$

Hence the domain of f is $D_f = [-3, 7]$.

(b) In the numerator of the function, $x-4 \not< 0 \Rightarrow x \not< 4$ and the denominator $x+1 \not\leq 0 \Rightarrow x \not\leq 1$. Combining the two relations, the domain of f is:

$$(-\infty,-1)\cup[4,\infty) \text{ or } [-1,-4)$$

Example 05: Let the function f be defined by $f(x) = 5x^2 - 4x + 8$. Determine:

(a) f(0) **(b) f(2)** **(c) f(-3)** **(d) f(x+1).**

Solution:

(a) At $x=0,\ f(0)=5(0)^2-4(0)+8=8$

(b) At $x=2,\ f(2)=5(2)^2-4(2)+8=20$

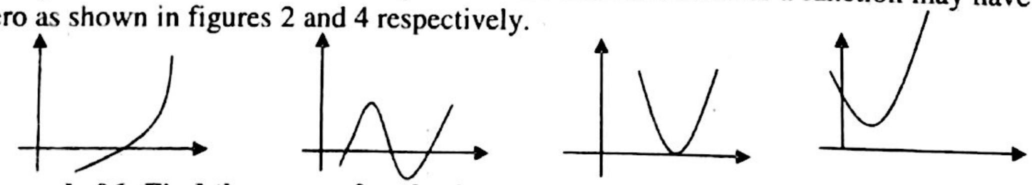
(c) At $x=-3,\ f(-3)=5(-3)^2-4(-3)+8=65$

(d) At $x=x+1,\ f(x+1)=5(x+1)^2-4(x+1)+8=5x^2+6x+9$

Zeros of a Function

A zero of a function f is any real number x for which f (x) = 0. Geometrically, it is a point where the function crosses or touches the x-axis. This is shown in the figures below:

It may be noted that a function may have more than one zeros or a function may have no zero as shown in figures 2 and 4 respectively.



Example 06: Find the zeros of each of the following functions:

(a) $f(x) = 5x - 20$ **(b) $f(x) = x^2 + 5x - 14$**

Solution: (a) Given f(x) = 5x – 20. Now if f(x) = 0 ➔ 5x – 20 = 0 ➔ x = 4. Thus the only zero of this function is 4.

(b) Given $f(x) = x^2 + 5x - 14$. If $f(x) = 0$ ➔ $x^2 + 5x - 14 = 0$. This is a quadratic equation. Solving, we get: $x = -7$ or $x = 2$. Thus given function has two zeros –7 and 2.

Types of Functions

By now, hopefully the readers have become familiar with the concept of function. We now present different types of functions which are very important for our further work.

1. Surjective Function: A function f from set X to set Y is said to be Surjective function if $R_f = Y$. Surjective function is also known as **onto function**. For example, the function $f(x) = x^3$ from R to R is onto, whereas the function $f(x) = x^2$ is not onto because $Y \neq R$..

2. Injective Function: A function f from set X to set Y is said to be an Injective if distinct elements of D_f have distinct images, that is, if for all $x_1, x_2 \in D_f$:

$$x_1 = x_2 \quad f(x_1) \quad f(x_2)$$

An Injective function is also known as **One–One function**. For example, the function $y = f(x) = x^3$ from R to R is one–one because different values of x have different images in f(x).

The function $f(x) = x^2$ from R to R on the other hand, is not one–one because different values of x have the same image. For if x = 1 then y = 1, if x = -1 then y = 1. Thus, for two different values of x, y has the same value. Hence this is not a 1-1 function.

3. Bijective Function: A function f from R to R is said to be a Bijective function if it is both Injective and Surjective. A Bijective function is also known as **one–to–one correspondence**. For example, the function $f(x) = x^3$ is Bijective function.

4. Even and Odd Functions

A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)$ and is called odd function if $f(-x) = -f(x)$. If a function does not satisfy these conditions, it is said to be neither even nor odd function.

For example, the function $f(x) = x^2 + 1$ is even function, for

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$$

The function $f(x) = x^3 + x$ is odd function because

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x).$$

The function $f(x) = x^3 - x$ is neither even nor odd function. Readers may verify it.

5. Inverse of a Function

Let $y = f(x)$ be a function of x. We define inverse function as: $x = f^{-1}(y)$.

For example if: $y = (x + 2)/(x - 7)$ ➔ $y(x - 7) = (x + 2)$

➔ $yx - x = 2 + 7y$ ➔ $x(y - 1) = (2 + 7y)$

➔ $x = (2 + 7y)/(y - 1)$ ➔ $x = f^{-1}(y) = (2 + 7y)/(y - 1)$

Example 07: If $f(x) = (x + 2)/(x - 7)$ find $f^{-1}(x)$ and hence $f^{-1}(3)$.

Solution: Let $y = f(x) = (x + 2)/(x - 7)$ then

$f^{-1}(y) = (2 + 7y)/(y - 1)$ as shown above. ➔ $f^{-1}(x) = (2 + 7x)/(x - 1)$

➔ $f^{-1}(3) = (2 + 21)/(3 - 1) = 23/2$

REMARK: It may be noted that if $y = f(x)$ be a function of x then its inverse $x = f^{-1}(y)$ may or may not be a function. $f^{-1}(y)$ is a function only if f(x) is both 1-1 and onto, that is; if $y = f(x)$ is bijective function then it's inverse $x = f^{-1}(y)$ is a function and moreover, the resultant inverse function is also bijective function.

Let us take an example:

(i) Consider $y = x + 5$. Since this function is bijective function hence, its inverse $x = y - 5$ is also a function. In fact $x = y - 5$ is bijective function.

(ii) Now consider the function $y = x^2$. It's inverse is $x = \pm\sqrt{y}$ is not a function because for one value of y there are exactly two values of x.

6. Composite Function

If

and $f : A \longrightarrow B$

$g : B \longrightarrow C$

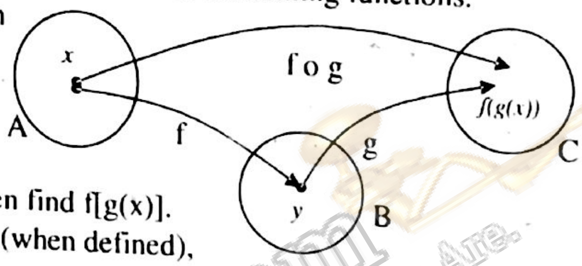
are functions of an independent variable x. Then composite function of f and g is denoted by $h = f \circ g$ ("f circle g") and is a function from A to C, that is;

$$h : A \longrightarrow C$$

Composite function is defined as: $(f \circ g)(x) = f(g(x)) = h(x)$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which g(x) lies in the domain of f. In fact composition is another method for combining functions.

Formation of composite function is shown in the adjacent figure.



It shows that two functions can be composed when the range of the first lies in the domain of the second (fig).

To find (fog) (x) we first find g(x) and then find f[g(x)].

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding f(x) first and then g[f(x)]. The domain of $g \circ f$ is the set of numbers x in the domain of f such that f(x) lies in the domain of g.

REMARK: The functions fog and gof are usually not equal, i.e. fog $\neq$ gof.

Example 08: If $f(x) = \sqrt{x^3 - 3}$ and $g(x) = x^2 + 3$ **find**

(i) fog (ii) gof (iii) fof (iv) gog

Solution: (i) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3) = \sqrt{(x^2+3)^3 - 3} = \sqrt{x^6 + 9x^4 + 27x^2 + 24}$

(ii) $(g \circ f)(x) = g(f(x)) = f(\sqrt{x^3 - 3}) = (\sqrt{x^3 - 3})^2 + 3 = x^3 - 3 + 3 = x^3$.

(iii) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x^3 - 3}) = \left(\sqrt{(\sqrt{x^3 - 3})^3 - 3}\right)$.

(iv) $(g \circ g)(x) = g(g(x)) = g(x^2 + 3) = (x^2 + 3)^2 + 3 = x^4 + 6x^2 + 12$.

Observe that fog $\neq$ gof.

Algebra of Functions

Let f and g be given functions. The sum $f + g$, the difference $f - g$, the product $f \times g$ and the quotient f/g are functions defined by:

(i) $(f + g)(x) = f(x) + g(x), \quad \forall\, x \in D_f \cap D_g$

(ii) $(f - g)(x) = f(x) - g(x), \quad \forall\, x \in D_f \cap D_g$

(iii) $(fg)(x) = f(x)g(x), \quad \forall\, x \in D_f \cap D_g$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \forall\, x \in D_f \cap D_g,\ g(x) \neq 0$.

The reciprocal of the function f is denoted by $1/f$ and defined as

(v) $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}, \quad \forall\, x \in D_f$ where $f(x) \neq 0$.

(vi) $(cf)(x) = cf(x), \quad \forall\, x \in D_f,\ c \in R$.

Example 09: If f(x) = 2x – 1 and g(x) = x^2 + 1 where $x \in R$, find

(i) (f + g) **(ii) (f – g)** **(iii) fg** **(iv) 1/f**
(v) f/g **(vi) -3f** **(vii) f(x + 2)** **(viii) g(x – 3)**

Solution:

(i) $(f+g)(x) = f(x) + g(x) = 2x - 1 + x^2 + 1 = x^2 + 2x = x(x+2) \quad \forall\ x \in R$

(ii) $(f-g)(x) = f(x) - g(x) = (2x-1) - (x^2+1) = -x^2 + 2x - 2 \quad \forall\ x \in R$

(iii) $(fg)(x) = f(x) \times g(x) = (2x-1)(x^2+1) = 2x^3 - x^2 + 2x - 1 \quad \forall\ x \in R$

(iv) $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{2x-1} \quad \forall\ x \in R, x \neq 1/2$

(v) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x-1}{x^2+1}, \quad \forall\ x \in R$

(vi) $(-3f)(x) = -3f(x) = -3(2x-1) = -6x + 3, \forall\ x \in R$

(vii) $f(x+2) = 2(x+2) - 1 = 2x + 3\ \forall\ x \in R$

(viii) $g(x-3) = (x-3)^2 + 1 = x^2 - 6x + 9 + 1 = x^2 - 6x + 10\ \forall\ x \in R$

Graphs of Functions

A drawing that shows the relationship between two variables is called a graph. This idea was developed by a French mathematician **Rene Descartes**. A graph describes a function in visual form.

There are various types of graphs. For example, histograms and the pie chart are used to display numerical information in the form of graph which is simple and quickly understandable. Scatter diagrams may be used in analyzing the results of a scientific experiment.

In calculus, graphs are used to give a geometric representation of a function. Moreover, simultaneous equations can be solved by drawing the graphs of the equations and finding the points of intersection. Graphs are particularly helpful in the study of Analytical Geometry and Calculus. Graphs of some well-known functions are now presented below.

1. Linear Function

A function described by the equation: y = mx + c is called a linear function. In other words, a linear function of x is one, which contains no term in x of degree higher than the first. The general form of a linear function is

$$ax + by + c = 0$$

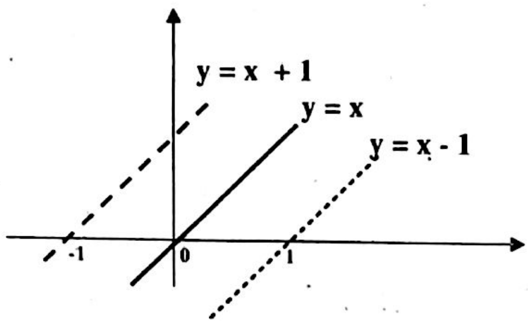
where x and y are variables, and a, b and c are constants. This function is called linear because its graph in the Cartesian coordinates is always a straight line.

A straight line is completely determined by two points; therefore to draw the graph of a straight line we locate at least two points in the plane. By joining the two points we obtain the required graph of the line. Let us take an example now.

Example 10: Draw the graph of the linear functions: y = x - 1, y = x, y = x + 1 on the same graph paper.

Solution: See the table below where tabular values of y = x – 1, y = x and y = x + 1 are given.

x	0	1
y = x – 1	-1	0
y = x	0	1
y = x + 1	1	2



REMARK: Look at the three graphs. Graph of $y = x - 1$ is shifted one unit to the right and graph of $y = x + 1$ is shifted one unit left. The shape of each graph is same.

Generally, for a given function $y = f(x)$, the shape of graphs of $f(x - a)$ and $f(x + a)$ will be same as that of $f(x)$ but graph of $f(x - a)$ is shifted `a` units to the right and that of $f(x + a)$ `a` units to the left of x-axis.

On the contrary, the graphs of $f(x) - a$ and $f(x) + a$ will have the same shape as that of $f(x)$ but they will be shifted `a` units down and up on the y-axis respectively, where $a > 0$.

2. Quadratic Function

A function defined by the equation $y = ax^2 + bx + c$, $a \neq 0$ where a, b, c are constants, is called a quadratic function. This equation always represents a parabola. The graph of parabola will have one of the shapes as shown below:

when $a < 0$ when $a < 0$

Similarly the equation $x = ay^2 + by + c$, where $a \neq 0$ represents the parabola having one of the following shapes:

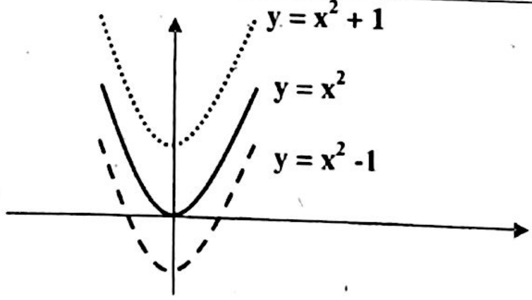
when $a > 0$ when $a < 0$

Thus, if one of the variables in the equation appears with single power and the other in its square form, then the graph is a parabola.

Example 11: Draw the graphs of $y = x^2 - 1$, $y = x^2$ and $y = x^2 + 1$ on the same graph paper.

Solution: See the table below where tabular values of $y = x^2 - 1$, $y = x^2$ and $y = x^2 + 1$ are given.

x	-1	0	1
$y = x^2 - 1$	0	-1	0
$y = x^2$	1	0	1
$y = x^2 + 1$	2	1	2

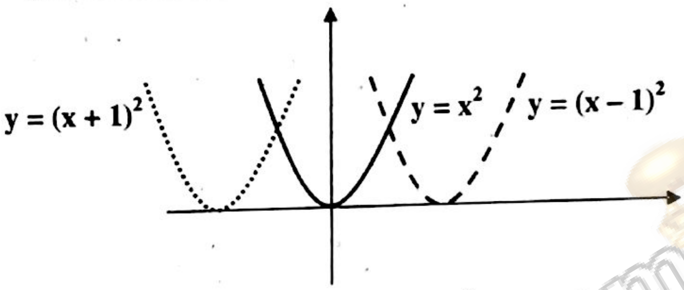


Observe that graphs of $y = x^2 - 1$ and $y = x^2 + 1$ are similar to the graph of $y = x^2$ but are shifted one unit down and one unit up respectively on the y-axis.

Example 12: Draw the graphs of $y = (x-1)^2$, $y = x^2$ and $y = (x+1)^2$ on the same graph paper.

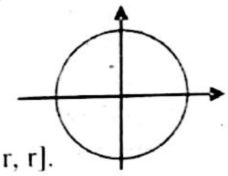
Solution: See the table below where tabular values of $y = x^2 - 1$, $y = x^2$ and $y = x^2 + 1$ are given.

x	-1	0	1
$y = (x-1)^2$	4	1	0
$y = x^2$	1	0	1
$y = (x+1)^2$	0	1	4



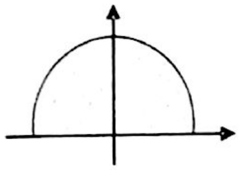
Observe that graphs of $y = (x-1)^2$ and $y = (x+1)^2$ are similar to the graph of $y = x^2$ but are shifted one unit right and one left respectively on the x-axis.

3. The Circle



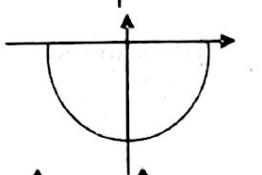
The equation of a circle with radius `r` and center (0, 0) is $x^2 + y^2 = r^2$ so that $y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$

This equation does not represent a function because we see that for one value of x, there are two values of y. Moreover, $D_f = [-r, r]$.

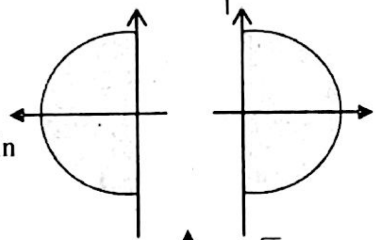


(i) $y = \sqrt{r^2 - x^2}$ is the upper semi – circle and the equation $y = -\sqrt{r^2 - x^2}$ represents the lower semi – circle.

Both $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ are functions and each one is called the branch of circle $x^2 + y^2 = r^2$.



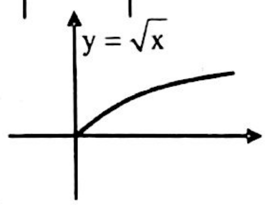
(ii) The equation $x^2 + y^2 = r^2$ can also be written as $x^2 = -y^2 + r^2$ so that $x = \pm\sqrt{r^2 - y^2}$. Since x is positive on the right of y – axis, therefore graph of $x = \sqrt{r^2 - y^2}$ is the right semi – circle.



Similarly $x = -\sqrt{r^2 - y^2}$ represents the left semi–circle.

Remark: The circle $x^2 + y^2 = 1$ whose center is at the origin and the radius is 1, is called the **unit circle.**

4. Square Root Function



Square root function is defined by $f(x) = \sqrt{x}, x \geq 0$.

The domain of the square root function consists of all nonnegative real numbers, that is, $(x \geq 0)$. because square root of a negative number is not a real number. Graph of this function is shown here.

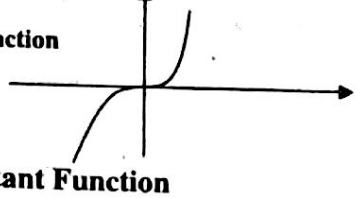
5. Cube Function

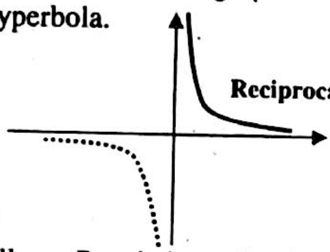
The Cube function is defined as $f(x) = x^3,\ x \in R$. The domain of the cube function consists of all the real numbers. The graph is shown as under.

6. Reciprocal Function

The reciprocal function is defined as: $f(x) = 1/x, x \neq 0$. The graph of reciprocal function is shown here and is known as rectangular hyperbola.

Cube Function

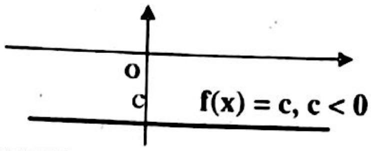


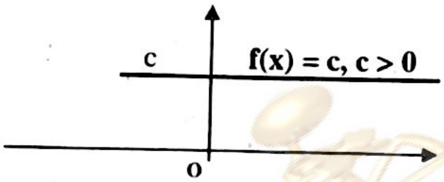


Reciprocal Function

7. Constant Function

Let $f : R \to R$ be defined by $f(x) = c$ for all $x \in R$, c being a fixed real number. Such function is known as constant function.

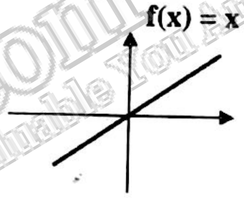




REMARK: (i) Function $f(x) = c$ is a straight line parallel to x – axis (ii) $f(x) = 0$ represents the x–axis.

8. Identity Function

A function $f : R \to R$ defined by $f(x) = x$ for all $x \in R$ is called an identity function. Its graph is a straight line passing through through the origin and making angle of $45°$ with the x and y–axes. Here $D_f = R$.



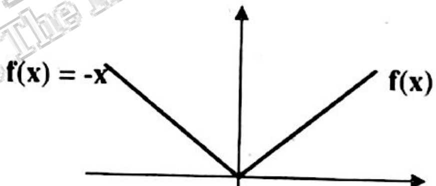
9. The Absolute Function

A function $f : R \to R$ defined by

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x > 0 \end{cases}$$

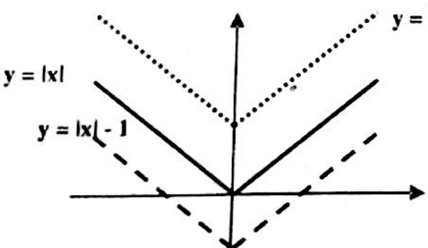
is called the absolute function.

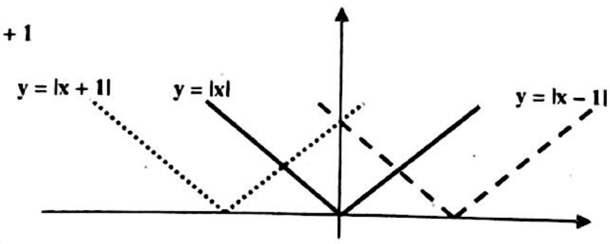
The graph of f consists of parts of the lines $f(x) = x$ and $f(x) = -x$ above the x – axis.



It may be noted that this function is usually denoted by $f(x) = |x|$. Hence, it is also known as modulus function.

REMARK: The graphs of $y = |x| - 1$, $y = |x|$, and $y = |x| + 1$ and those of $y = |x - 1|$, $y = |x|$ and $y = |x + 1|$ are shown as under. You may observe the shifting along y-axis and along x-axis respectively.



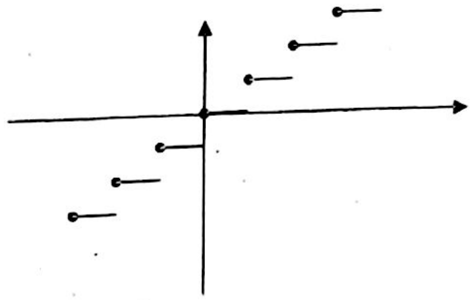


10. The Bracket Function

(i) Greatest Integer Function: A function whose value at any number x is the greatest integer less than or equal to x, is called 'Greatest Integer Function' and is usually denoted by $f(x) = \lfloor x \rfloor$. It is also known as **"floor function".**

The graph is shown here.

$$f(x)=0, \quad 0 \le x < 1$$
$$=1, \quad 1 \le x < 2$$
$$=2, \quad 2 \le x < 3$$
$$=3, \quad 3 \le x < 4$$
....
$$=-1, \quad -1 \le x < 0$$
$$=-2, \quad -2 \le x < -1$$
$$=-3, \quad -3 \le x < -2$$

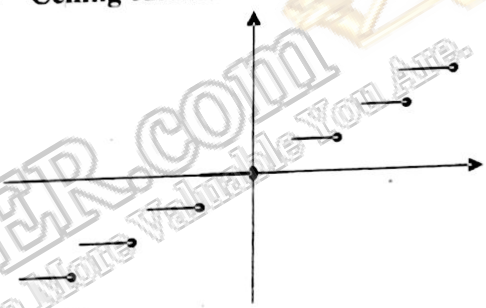


Notice that, $f(2.4)=\lfloor 2.4 \rfloor = 2$, $f(1.9)=\lfloor 1.9 \rfloor = 1$, $f(-0.3)=\lfloor -0.3 \rfloor = -1$, $f(-1.2)=\lfloor -1.2 \rfloor = -2$, $f(2)=\lfloor 2 \rfloor = 2$, $f(-2)=\lfloor -2 \rfloor = -2$ and so on. It may be noted that $D_f = R$.

(ii) The Least Integer Function

A function whose value at any number x is the smallest integer greater than or equal to x, denoted by $f(x)=\lceil x \rceil$. It is also known as " **Ceiling function**". To draw the graph, we find the points given below:

$f(x) = -1, -2 < x \le -1$
$= 0, -1 < x \le 0$
$= 1, 0 < x \le 1$
$= 2, 1 < x \le 2$
$= 3, 2 < x \le 3$



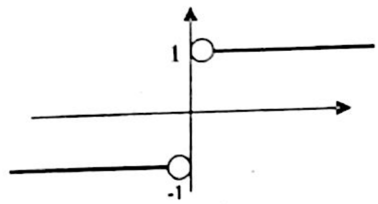
It may be noted that:

$f(3)=\lceil 3 \rceil = 3$, $f(-5)=\lceil -5 \rceil = -5$, $f(2.4)=\lceil 2.4 \rceil = 3$, $f(1.9)=\lceil 1.9 \rceil = 2$, $f(-0.3)=\lceil -0.3 \rceil = 0$, $f(-1.2)=\lceil -1.2 \rceil = -1$ and so on. Also $D_f = R$.

11. The Sign Function:

The sign function is defined as follows:

$$\operatorname{sgn}(x)=\begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$$



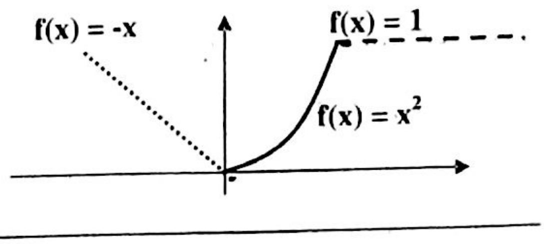
The graph of sign function is shown above. For $x \ne 0$ we sgn(x) function is also defined as:

$$\operatorname{sgn}(x)=\frac{|x|}{x}=\frac{x}{|x|}, \qquad D_f = R-\{0\}.$$

12. Piecewise Function

Sometimes a function uses different formulas on different parts of its domain. Such types of functions are called piecewise functions. For example, consider a function defined as under.

$$f(x)=\begin{cases} -x, & x<0 \\ x^2, & 0 \le x \le 1 \\ 1, & x>1 \end{cases}$$



is defined on the entire real line but has values given by different formulas depending on the value of x.

The graph of this function is drawn by applying different formulas as given in the function f(x).

REMARK: Applications of Functions and piecewise functions will be presented in the next section.

13. Polynomial Function

An expression of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is known as a polynomial of degree n. Hence, the function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is known as a polynomial function. The domain of a polynomial function is a set of all real numbers.

REMARK: A function is called an **"Algebraic Function"** function if it can be expressed as the sum, difference, products, quotients, powers or roots of polynomials.

For example, $f(x) = 2x^3 + 5x - 7$ is polynomial function and $g(x) = (x^2 + 1)/(x-2)$, $h(x) = \sqrt{x^2 - 2x + 5}$ and $k(x) = x(x^2 + 5) + \sqrt{2x + 5}$ are algebraic functions.

14. Transcendental Function

Functions that are not algebraic are known as transcendental functions. The combination of algebraic and transcendental functions is also a transcendental function. Such functions contain any trigonometric, inverse trigonometric, exponential or logarithmic functions. For example, the function:

$f(x) = x^3 + \sqrt{x-6} + \cos x - e^x + \log x$ is a transcendental function.

15. Bounded Functions

A function f(x) from R to R is said to be **bounded** if range of f is bounded otherwise it is unbounded function. For example, $f(x) = \sqrt{4 - x^2}$ is bounded function because its range is [-2, 2]. This function represents the upper half of the circle centered at origin and radius 2. Graph is shown on page 10.

The functions $f(x) = x^2$, $g((x) = x + 1$ are unbounded because the range of f(x) is the set of non-negative real numbers and that of g(x) is the set of all real numbers. Both sets are unbounded hence f(x) and g(x) are unbounded functions.

16. Circular and Hyperbolic Functions

Circular Functions

Readers are familiar with trigonometric functions such as sin x, cos x, tan x etc. These functions are also known as circular functions because they are having a relation with a circle. For instance, cos x and sin x satisfy the equation $x^2 + y^2 = 1$ of a circle, therefore these functions are called circular functions. In this section we shall look at these functions from different angle. However, we have to make some assumptions which we shall prove later on.

The trigonometric function sin x, cos x and exponential function e^x are the functions, which can be expressed in infinite series of non–negative increasing powers of x. Such series are called "power series" and are given by:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (2)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \qquad (3)$$

Replacing x by ix in (3) where i(iota) is an imaginary unit and $i = \sqrt{-1}$, such that $i^2 = -1$, we get

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Using the first two series, we get

$$e^{ix} = \cos x + i \sin x \qquad (4)$$

This identity was established by Euler and therefore is known as "**Euler's Identity**" or "**Euler's Formula**". Replacing x by –x in (4), we get

$$e^{-ix} = \ = \cos x - i \sin x \qquad (5)$$

NOTE: sin (-x) = - sin x and cos (-x) = cos x

Adding (4) and (5), we get: $e^{ix} + e^{-ix} = 2\cos x \Rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

Subtracting (5) from (4), we get: $e^{ix} - e^{-ix} = 2i \sin x \Rightarrow \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

Now by definition,

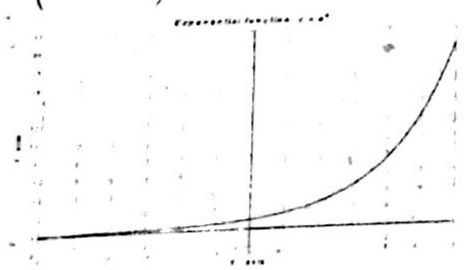
$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(e^{ix} - e^{-ix}\right)/2i}{\left(e^{ix} + e^{-ix}\right)/2} = \frac{e^{ix} - e^{-ix}}{i\left(e^{ix} + e^{-ix}\right)}, \cot x = \frac{1}{\tan x} = \frac{i\left(e^{ix} + e^{-ix}\right)}{e^{ix} - e^{-ix}},$$

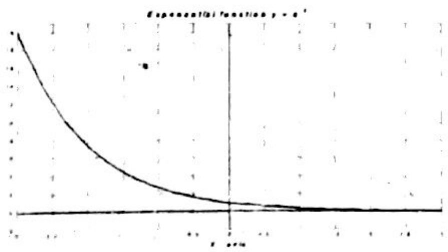
$$\sec x = \frac{1}{\cos x} = \frac{2}{e^{ix} + e^{-ix}}, \csc x = \frac{1}{\sin x} = \frac{2i}{e^{ix} - e^{-ix}}.$$

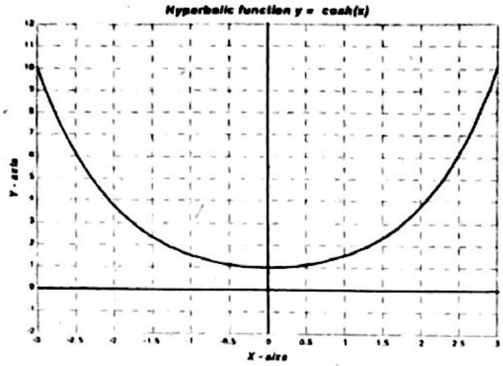
Hence, we have 'six circular functions' given below:

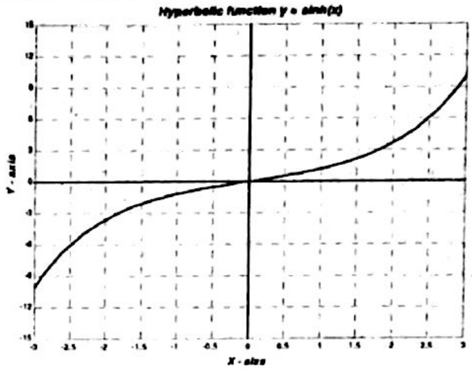
(ii) Hyperbolic Functions

In the following figures, we have shown the graphs of the exponential functions e^x and e^{-x} respectively. Moreover, the graphs of the curve $y = \left(e^x + e^{-x}\right)/2$ and that of $y = \left(e^x - e^{-x}\right)/2$ are also shown below:









It is found that these two functions have properties which in many respects are similar to those of y = cos x and y = sin x. It has been known that hyperbolic functions bear a similar relation to the hyperbola which the trigonometric or circular functions do to the circle. The function $y = (e^x + e^{-x})/2$ is called the hyperbolic cosine of x, and $y = (e^x - e^{-x})/2$ is called the hyperbolic sine of x.

These are abbreviated to cosh x and sinh x respectively. They are defined by the equations stated above, that is: $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = y = (e^x - e^{-x})/2$.

They describe the motions of waves in elastic solids, the shapes of hanging electric power lines, and the temperature distributions in metal cooling fins. There are four other hyperbolic functions. They are

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},\ \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

Hence we have six hyperbolic functions given below:

$\sinh x = y = (e^x - e^{-x})/2.$ $\quad \cosh x = (e^x + e^{-x})/2$

$\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ $\quad \coth x = (e^x + e^{-x})/(e^x - e^{-x})$

$\operatorname{sech} x = 2/(e^x + e^{-x})$ $\quad \operatorname{csch} x = 2/(e^x + e^{-x})$

REMARK: The curve of cosh x is an important one. It is called the catenary, and is the curve formed by a uniform flexible chain that hangs freely with its ends fixed. The graph of catenary is shown here.

Flexible wire with fixed ends

These functions can be expressed in the form of series, which are derived from the series for e^x. Since, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

Hence by addition and subtraction:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{and} \quad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Hyperbolic Identities

There is a close correspondence between formulae expressing relations between hyperbolic functions and similar relations between circular functions. Consider the following.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left\{(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right\} = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) = \frac{1}{4}(4) = 1$$

Thus, $$\cosh^2 x - \sinh^2 x = 1 \quad (1)$$

This is the basic relation of hyperbolic functions and is analogue of the trigonometric relation, $\cos^2 x + \sin^2 x = 1$.

Dividing both sides of (1) by $\cosh^2 x$, we get

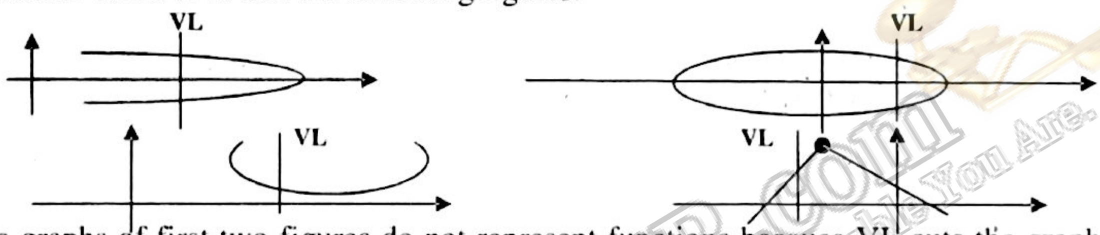
$$1 - \tanh^2 x = \text{sech}^2 x \quad (2)$$

Dividing both sides of (1) by $\sinh^2 x$, we get

$$\coth^2 x - 1 = \text{csch}^2 x \quad (3)$$

Vertical Line Test

The definition of a function says that for every x there is one y. The graphical interpretation of this idea is considered in the vertical line test. If a vertical line (VL) intersects a curve in two or more points, then the graph of the given curve does not represent a function. In this case, there would be two or more y values corresponding to a particular value of x. See the following figures:



The graphs of first two figures do not represent functions because VL cuts the graphs at two points, whereas, the last two graphs represent the function because VL cuts the graphs at only one point.

1.3 PHYSICAL APPLICATIONS OF FUNCTIONS

There is perhaps no field or area where functions are not used. In real life, where there is a relation between two variables, the application of function is must. These applications are found in social and natural sciences, engineering and medical sciences, etc. Applications of functions are also known as mathematical modeling of functions.

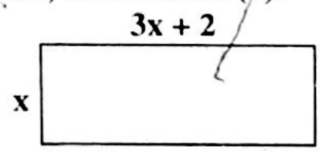
Case Studies (Applied Problems and Simple Mathematical Modeling)

Example 01: A rectangular fence is to be constructed so that its length is 3x + 2 and its width is x meters. If P is the function that gives perimeter, determine P(x)?

Solution: Perimeter is the sum of the sides of any polygon.

Now for rectangle as shown in the figure,

P(x) = 2(3x + 2) + 2x = 8x + 4 meters.



Example 02: A colony of bacteria is placed into a growth inhabiting environment. The number of bacteria present at any time t (in hours) is given by $n(t) = 1000 + 20t + t^2$. Find: n(0), n(1) and n(10). Interpret your results.

Solution: n(0) = 1000 + 0 + 0 = 1000. This means that at the beginning (t = 0) the number of bacterial in the colony is 1000.

n(1) = 1000 + 20 + 1 = 1021. This means that after one hour, number of bacteria in the bacterial colony is 1021.

$n(10) = 1000 + 20(10) + (10)^2 = 1000 + 200 + 100 = 1300$. This means that number of bacteria in the colony after 10 hours is 1300.

Example 03: When a car is moving at x miles per hour and the driver decides to slam on the brakes, the car will travel $x + 0.5x^2$ ft. If car travels 175 ft after the driver decides to stop, how fast was car moving?

Solution: The answer to this problem is very simple. Equating both conditions, we get:

$$x + 0.5x^2 = 175$$

Multiplying by 2, we get: $x^2 + 2x - 350 = 0.$

This is quadratic equation. Solving, we obtain: x = 17.8 ⊔ 18 miles/h.

Example 04: A person weighing 150 pounds on earth has weight given by:

$$w(d) = \frac{2,400,000,000}{(4,000 + d)^2}$$

miles above the earth surface.

a. **Find w(0) and interpret this?**
b. **How much this person weigh while flying in air plane at 29,000 feet?**
c. **An astronaut orbits the earth at an average of 80 miles above the surface. If he weighs 150 pounds on earth, how much does he weigh while in orbit?**

Solution: (a) w(0) = (2, 400, 000, 000)/(4, 000)^2 = 150. This means that weight of a person on the earth is 150 lbs.

(b) d = 29, 000 ft = 29, 000/(5280) = 5.5 miles. **(NOTE: 1 mile = 5280 ft)**

Putting this in given equation, we get:

w = (2, 400, 000, 000)/(4005.5)^2 = 149.6

This means the weight of person at the height of 29, 000 ft above the ground is 149.6 lbs.

(c) Putting d = 80 in the formula, we get:

w = (2, 400, 000, 000)/(4080)^2 = 144.2 lbs

Example 05: A number of degrees d in each interior angle of a region of polygon of n sides are: $d = (180n - 360)/n$

Use this formula to compute the number of degrees for a polynomial of sides 3, 4, 5 and 6.

Solution: If n = 3, d = [180(3) - 360]/3 = 60° → Each angle of equilateral triangle is 60°.
If n = 4, d = [180(4) - 360]/4 = 90° → Each angle of square is 90°.
If n = 5, d = [180(5) - 360]/5 = 108° → Each angle of regular pentagon is 108°.
If n = 6, d = [180(6) - 360]/5 = 144° → Each angle of regular hexagon is 144°.

REMARK: This formula is valid only for regular polygons.

Example 06: One of the methods to determine the children's dosage D of medicine is given by D(c) = [c – 1]a/24 where, c is the child age and `a` is the dosage of adult. If a child is 8 year old, what is his dosage if adult dosage is 400 mg? Interpret the result.

Solution: Putting c = 8 and a = 400, we get: D = (8 – 1).400/24 = 116.7 mg. Thus according to formula given, if the adult dosage is 400 mg and the child is 8 year old then his dosage will be 116.7 mg.

Example 07: The following formula is used to know the depreciated value of an item: D = C – [(C – S)t]/n where t is time in years, D the depreciated value after t years, C is the original cost, n is the useful life in years and S is the scrape value (Resale value). What will be the depreciated value of the machine after 8 years when it was purchased for 3400 dollars and its present value (scrape value) is 400 dollars and the useful time is 15 years?

Solution: Here, C = $3400, S = $400, n = 15 years, t = 8. Thus,

D = 3400 - [8(3400 – 400)]/15 = $1800

Thus depreciated value of the machine after 8 years was $1800.

REMARK: In this problem, we are told that after 15 years the value of machine is $400 and we are asked that under this condition what was the value of machine after 8 years of its purchase.

Example 08: Consider the following data on length y and age x of an alligator.

x	1	2
y	1.4	2.6

Assume the relationship between two variables is linear (y = mx + c), find

(a) The length of alligator when it is 4 years old

(b) The age when alligator is 6.8 feet long

Solution: From geometry we know that slope of a line is $m = (y_2 - y_1)/(x_2 - x_1)$. Using the above information, m = (2.6 – 1.4)/(2 – 1) = 1.2. Thus relation between length and age of alligator is:

$y - y_1 = m(x - x_1)$ ➔ $y - 1.4 = 1.2(x - 1)$ ➔ $y = 1.2x + 0.2$

(a) If alligator is 4 year old then its length y = 1.2(4) + 0.2 = 5 feet.

(b) If alligator is 6.8 feet long then: 6.8 = 1.2x + 0.2 ➔ x = 5 ½ years.

Example 09: Workers at a fast food restaurant earn $5 per hour for the first 40 hours in a week and then $7.5 per hour for additional hours. Let x be the number of hours worked in a week, write a two piece function P that describes a worker's pay. What would be the pay if a worker works for (a) 35 hours (b) 45 hours.

Solution: If a worker works for 0 up to 40 hours, he will be paid $5 per hour. Hence his pay is a function of x (hours), which is

$$P(x) = 5x \quad \text{for } 0 \le x \le 40$$

When x is greater than 40, the worker makes $5 per hour for 40 hours ($200 total) plus $7.5 per hour for each extra hour over 40. Now the extra hours are (x – 40) and then earning would be {200 + 7.5(x – 40)} dollars. After simplification, we have:

$$P(x) = 7.5x - 100 \text{ for } x > 40$$

Thus, the two – piece function P is given by the formula

$$P(x) = \begin{cases} 5x, & \text{for } 0 \le x \le 40 \\ 7.5x - 100, & \text{for } x > 40 \end{cases}$$

Now P(35) = 5(35) = $175, and P(45) = 7.5(45) – 100 = $237.5

Example 10: The monthly charge for water in a small town is given by

$$f(x) = \begin{cases} 18, & \text{for } 0 \le x \le 20 \\ 18 + 0.1(x - 20) & \text{for } x > 20 \end{cases}$$

where x is in hundreds of gallons and f is in dollars. Find the monthly charge for each of the following usages.

(i) 30 gallons (ii) 3000 gallons (iii) 4000 gallons

Solution: (i) Since, x is in hundreds of gallons, hence 1 unit of gallon will be 1/100 = 0.01. Thus 30 gallons will be equivalent to 0.30 units. Now, according to the given domain: f(0.3) = $18

(ii) Since x is in hundreds of gallons, 3000 gallons will be equivalent to 3000/100 = 30 of units. Now, according to given domain:

$$f(30) = 18 + 0.1(30 - 20) = \$19$$

(iii) Since x is in hundreds of gallons, 4000 gallons will be equivalent to 4000/100 = 40 of units. Now, according to given domain:

$$f(40) = 18 + 0.1(40 - 20) = \$20$$

REMARK: 1 Gallon = 4.546 liters.

Functions in Economics

Functions that provide information about cost, revenue, and profit can be of great value to management. This section offers an introduction to the cost function (C) revenue function (R) and profit function (P). We begin by establishing the notation for three important types of functions. Using x for the number of units produced and/or sold, we have

Cost, Revenue, Profit

$C(x) \equiv$ The total cost of producing x units.

$R(x) \equiv$ The total revenue from the sale of x units.

$P(x) \equiv$ The total profit from the production and sale of x units.

Example 11: Assume that the cost of producing x computer chips is

$$C(x) = 0.4x^2 + 7x + 95 \text{ \$}$$

(a) Find the cost of producing 20 chips

(b) Determine the cost of producing the 20th chip

(c) Determine the cost of producing no chip. Interpret the result

Solution: (a) The cost of producing 20 items is $C(20) = 0.4(20)^2 + 7(20) + 95 = \$395$.

(b) The cost of producing the 20th chip = C(20) – C(19) = 395 – 372.4 = 22.6 dollars.

(c) C(0) = \$95. This means if no chip is produced; the cost is \$95. This is also known as fixed cost. It includes the cost of the machine, purchase of equipments and raw material, design and other expenses that exist before the production starts.

A profit function P(x) is sometimes given directly, however, it may be determined using revenue and cost function as under:

$$\text{Profit} = \text{Revenue} - \text{Cost} \quad \textbf{OR} \quad P(x) = R(x) - C(x)$$

Example 12: It costs a manufacturer $C(x) = 0.4x^2 + 7x + 95$ dollars to produce x computer chips. They can be sold at \$40 each; that is, revenue from the sale of x chips is R(x) = 40x dollars.

(a) Determine the profit function

(b) What is the profit on the manufacture and sale of 25 chips?

(c) What is the profit on the manufacture and sale of 25th chip?

(d) What is the profit on the manufaoture and sale of 2 chips?

Solution: (a) Using P(x) = R(x) – C(x), we have

$$P(x) = 40x - (0.4x^2 + 7x + 95) = -0.4x^2 + 33x - 95\$$$

(b) $P(25) = -0.4(25)^2 + 33(25) - 95 = \$480$.

This means on manufacturing and sale of 25 chips, the profit will be \$480.

(c) The profit on the sale of 25th chip = P(25) – P(24) = 480 – 466.6 = \$13.4

(d) P(2) = -0.4(4) +33(2) – 95 = -30.6

The negative sign shows that there is a loss of 30.6 dollars. This means that the company would loss \$30.6 on the production and sale of only 2 chips.

Break-Even Point (BEP)

When the production level x is such that revenue and cost functions become equal that when R(x) = C(x) then profit is zero. Such value of x is known as break-even point. This indicates that company is running with "No Loss and No Gain".

Geometrically, BEP is that value of x where the graphs of C(x) and R(x) intersect each other. This is depicted in the following figure. This phenomenon is very much important for the managers to see that how many items should be produced to have a profit.

Example 13: The management of a publishing company informs the marketing department that the profit function is P(x) = 0.08x – 15, 200 dollars where x is the number of items that are sold of sales.

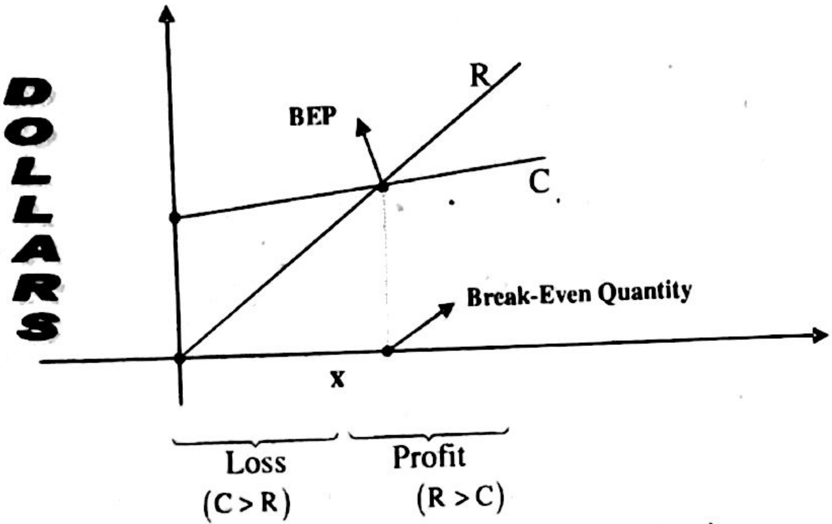
(a) How would management react to sales of 100,000 items?

(b) How many dollars of sales are needed to break – even?

Solution: (a) When $x = 100,000$, we have,

$$P(100,000) = 0.08(100,000) - 15,200 = -7,200 \text{ dollars}$$

This indicates that the management will bear a loss of \$7200 at the sale of 100,000 items. Thus marketing department will suggest the management to increase the production level.



(b) For break – even, point, the profit must be zero. That is,

$$-.08x - 15,200 = 0 \Rightarrow x = 190,000$$

Thus, 190,000 of items are to be produced and sold for the break–even point. In other words, the management must produce and sale at least 190,000 items to save the company from any loss.

Example 14: (Price and Demand) Assume that for some product, the equation $p = 80 - 0.2x$ dollars gives the relationship between the price per unit p and the quantity x demanded. How many items are demanded if the price is set at (a) \$70 (b) \$65? Interpret the two results.

Solution: (a) If p = \$70, then: $70 = 80 - 0.2x \Rightarrow x = 50$

(b) If p = \$ 65, then: $65 = 80 - 0.2x \Rightarrow x = 75$

This shows that if price is set at high side the demand will be less and if the price lies on the low side, demand will be at the high level.

Example 15: If price of one unit is $p = 80 - 0.2x$ what will be the revenue function if x items are sold? Find R(90).

Solution: If R is the revenue function and p is the price of one unit then revenue from the sale of x units is: $R(x) = px = (80 - 0.2x)x = 80x - 0.2x^2$ dollars.

Then, $R(90) = 80(90) - 0.2(90)^2 = 5580$ dollars.

WORKSHEET 01

1: Draw the graphs of the following functions. Also mention the domain and range of these functions.

(a) $y = 2x + 7$ (b) $y = 2x^2 + 1$ (c) $y = (x^2 + 1)/(x - 1)$

2. Solve each of the following inequalities:

(a) $|2x + 5| > |2 - 5x|$ (b) $|x| + |x - 1| > 1$ (c) $12x^2 - 25x + 12 > 0$

(d) $|x^2 + x + 1| > 1$ (e) $x^{-2} - 4x^{-1} + 4 > 0$ (f) $2x/(x + 2) \geq x/(x - 2)$

3. If $f(x) = \sqrt{x^2 - 1}$ and $g(x) = \frac{1}{\sqrt{4 - x^2}}$, show that fog$\neq$ gof.

4. In 1998, a patient paid \$300 per day for a semiprivate hospital room and \$1500 for an appendectomy operation. Express the total amount for an appendectomy as a function of number of days of hospital confinement.

5. In some cities you can rent a car for \$18 per day and \$ 0.20 per miles.

a. Find the cost of renting the car for one day and driving 200 miles.

b. If the car is rented for one day then express the total rental expenses as a function of the number x miles driven.

6. Suppose the longer side of a rectangle has twice the length of shorter side, and if x is length of shorter side, express the perimeter of the rectangle as a function of x.

7. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a commercial customer is given by the following function:

$$C(x)=\begin{cases}7.52+0.1079x, & 0\le x\le 5\\ 19.22+0.1079x, & 5<x\le 750\\ 20.795+0.1058x, & 750<x\le 1500\\ 131.345+0.0321x, & x>1500\end{cases}$$

Find the monthly charge for the following usages.
(a) 5 KWH (b) 6 KWH (c) 3000 KWH.

8. The pressure P of a certain gas is related to volume V according to: P = 100/V
(a) Is 0 in the domain of this function?
(b) What are P(100) and P(50)?
(c) As volume decreases, what happens to pressure?

9. The total cost of producing a product is given by
$$C(x) = 300x + 0.1x^2 + 1{,}200 \text{ dollars}$$
where x represents the number of units produced.
(a) What is the total cost of producing 10 units?
(b) What is the average cost per unit when 10 units are produced?

10. A cigar box distributor's revenue is R(x) = 1.35 x dollars. Where x is the number of boxes sold.
(a) How much revenue is obtained from selling 5 boxes?
(b) How much revenue is obtained from the sale of 5th box?
(c) How much revenue is obtained from the sale of 8th box?

11. It costs a TV manufacturer $C(x) = 0.1x^2 + 150x + 1,000$ dollars to produce x TV sets. The revenue from the sale of x TV sets is R(x) = 280x dollars.
(a) Determine the profit function.
(b) What is the profit on the manufacture and sale of 50 TV sets?

12. The cost on producing x radios is $C(x) = 0.4x^2 + 7x + 95$ \$. The revenue received is R(x) = 40x \$. What is the profit function? Find P(24) and P(25). What is the profit on the sale of 25th radio?

13. A psychologist needs volunteers for an experiment. She offers to pay \$ 8 per hour for volunteer who works up to 5 hours. Those who work more than 5 hours are paid \$10 per hour for additional hour.
(a) Write down the function that represents the volunteer's pay V(x), where x represents the hours worked. (b) Also find V(5), V(10) and V(15).

14. If p = 0.01x + 19 dollars is the price of one jacket and each jacket is sold for \$80, determine the profit function and then find P(2) and (50). What is BEP?

15. Sketch the graph of following piecewise functions.

(a) $f(x)=\begin{cases}2 & \text{for } x\le 0\\ -x & \text{for } x>0\end{cases}$ (b) $f(x)=\begin{cases}x & \text{for } x<0\\ x^2 & \text{for } x\ge 0\end{cases}$

(c) $f(x)=\begin{cases}x & \text{for } x\le 1\\ \sqrt{x} & \text{for } x>1\end{cases}$ (d) $f(x)=\begin{cases}|x| & \text{for } x\le 0\\ 2x & \text{for } x>0\end{cases}$

16. A motorbike was purchased 2 years ago for Rs. 50, 000 is now worth Rs. 25, 000. Find the linear relationship between its value (y) and the life (x) in years.

(a) How many years from the time of purchase will it be before motorbike is worth Rs. 35, 000.
(b) What would be its cost after 10 years?
17. A house was purchased 5 years ago for Rs. 5 million is now worth Rs. 8 million. Find the linear relationship between its value (y) and the life (x) in years.
(a) How many years from the time of purchase will it be before house is worth Rs. 6 millions.
(b) What would be its cost after 8 years?
18. State which of the following functions is even or odd.
(a) $f(x) = x^4 + x^2 - 1$
(b) $g(x) = x^3 + \sin x + 1$
(c) $h(x) = x^3 - x$
(d) $k(x) = |x| + \cos x - 2$
19. What is the domain and range of the following functions?
(i) $f(x) = 1/(x^2 - 1)$
(ii) $g(x) = (x + 1)/(2x - 1)$
(iii) $h(x) = 1/\sqrt{x^2 + 9}$
(iv) $k(x) = \dfrac{1}{\sqrt{16 - x^2}}$
(v) $m(x) = \dfrac{1}{\sqrt{x^2 - 16}}$
(vi) $n(x) = \sqrt{16 - x^2}$
(vii) $p(x) = \sqrt{x^2 - 16}$
(viii) $q(x) = \sqrt{x^2 - 16}/\sqrt{x^2 - 9}$

CHAPTER TWO

LIMITS AND CONTINUITY

2.1 THE CONCEPT OF LIMIT

The concept of limit of a function is one of the fundamental ideas that distinguish CALCULUS from other branches of mathematics.

To understand the concept of limit, let us consider a circular disc with area of one unit. If we remove half of the disc, area of remaining part is 1/2. If we further remove half of the remaining disc, the area of the left over part is 1/4 or $1/2^2$. By further removing half of the remaining disc, the area of left over part will be 1/8 or $1/2^3$. If this process is continued, that is, at every step we remove half of the remaining disc; the area of left over part at the n^{th} stage will be $1/2^n$. Thus, we observe that there is always some portion left over, however small it might be, and the process will continue indefinitely. The area of the left over part gets smaller and smaller and ultimately it will decrease to zero. We describe this fact by saying that limit of the left over area is zero. More precisely, we say that the limit of sequence of numbers: $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$ is zero.

Before giving informal definition of limit, consider the function, $f(x) = (x^2 - 25)/(x - 5)$. This function is defined for all vales of x except 5. For If we substitute x = 5, we get $f(5) = (25 - 25)/(5 - 5) = 0/0$ which is a meaningless quantity. One could perhaps say here that why don't we cancel (x – 5) first and then put x = 5 to get f(5) which is equal to 10. There is a lapse in this argument as (x – 5) is zero when x = 5 and cancellation of zero factor is not allowed in Mathematics. Consequently, we cannot determine the value of f(x) at x = 5. But we do not leave the problem here. Instead, we try to evaluate the value of f(x) when x is very near to 5 and this will finally lead us to a value that would almost be the value of f(5). Thus we can evaluate f(x) at x = 4.99 or x =5.1. The technique is quite simple. Cancel (x-5) first (this step is perfectly legitimate as x is not equal to 5); then substitute the value of x. For example, f(4.99) = 4.99 + 5 = 9.99 and f(5.01) = 5. 01+ 5 = 10.01. We now write down some of the values given to x and the corresponding values acquired by f(x) in the following tables.

Table 1: x approaches 5 from left

x	4	4.7	4.9	4.99	4.999
f(x)	9	9.7	9.9	9.99	9.999

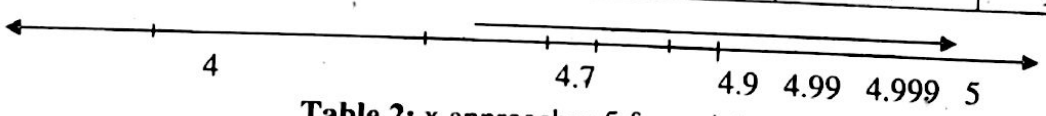
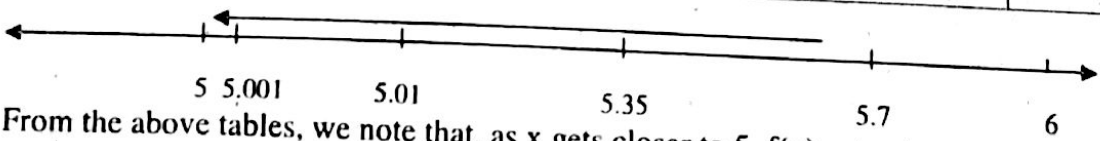


Table 2: x approaches 5 from right:

x	5.001	5.01	5.35	5.6	← 6
f(x)	10.001	10.01	10.35	10.7	11



From the above tables, we note that, as x gets closer to 5, f(x) gets closer to 10. In such case, we say that limit of f(x), as x approaches 5, is 10 and we write: $\lim_{x \to 5} f(x) = 10$.

This example leads us to an informal definition of limit.

Formal and Informal Definitions of Limit

Formal Definition

Let \`a\` be any real number and let f be a function from R to R which is defined for all x near to \`a\` with the possible exception of the point x = a. The function f is said to have a limit L (where L is real) as x approaches \`a\` if for every $\varepsilon > 0$, there exists a positive real number δ (usually depends on ε) such that

$$|f(x) - L| < \varepsilon \text{ wherever } \quad 0 < |x - a| < \delta.$$

In this case, we write: $\lim_{x \to a} f(x) = L$

Example 01: Prove, by definition, that $\lim_{x \to 3} 2x = 6$.

Solution: Here f(x) = 2x and L = 6 and a = 3. Thus by definition,

$$|f(x) - L| = |2x - 6| = 2\,|x - 3|.$$

Now let, $\quad |x - 3| < \delta \Rightarrow |f(x) - L| < 2\,\delta = \varepsilon$

This proves that: $\quad \lim_{x \to 3} 2x = 6$

Example 02: Prove, by definition, that $\lim_{x \to 2} \dfrac{x^2 + 2}{x + 1} = 2$

Solution: Here $f(x) = \dfrac{x^2 + 2}{x + 1}$ and L= 6/3 = 2 and a = 2. Thus by definition,

$$|f(x) - f(a)| = |f(x) - L| = \left|\frac{x^2 + 2}{x + 1} - 2\right| = \left|\frac{x^2 + 2 - 2x - 2}{x + 1}\right| = \left|\frac{x^2 - 2x}{x + 1}\right| = \left|\frac{x(x - 2)}{x + 1}\right|$$

$$= \left|\frac{x}{x + 1}\right| |x - 2|$$

Now let, $|x - 2| < \delta \Rightarrow |f(x) - L| = \left|\dfrac{x}{x + 1}\right| |x - 2| < \left|\dfrac{x}{x + 1}\right| \delta = \varepsilon$

This proves that: $\quad \lim_{x \to 3} \dfrac{x^2 + 2}{x + 1} = 2$

Informal Definition

Let a function be defined for all values of x except possibly for x = a where \`a\` is a real number. The function f is said to have the limit L (a real number) as x approaches \`a\`, if the value of f(x) can be made as close to L as we please by taking x sufficiently close to (but not equal to) \`a\`. In such case, we write: $\lim_{x \to a} f(x) = L$

Note that x may be close to \`a\` both from left and right of \`a\`.

Sometimes, $\lim_{x \to a} f(x)$ can be evaluated by calculating f(a). This holds, for example, whenever f(x) is an algebraic combination of polynomials and/or trigonometric functions for which f(a) is defined. For example,

(a) $\lim_{x \to 3} x = 3$ $\qquad$ (b) $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

(c) $\lim_{x \to 0} (1 + \cos x) = 1 + 1 = 2$ $\qquad$ (d) $\lim_{x \to -2} \dfrac{3x + 4}{x + 5} = \dfrac{-6 + 4}{-2 + 5} = -\dfrac{2}{3}$

Theorems on Limits

Theorem 1: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ (L and M are real numbers). Then

1. Sum Rule: $\lim_{x\to a}[f(x)+g(x)] = L+M = \lim_{x\to a}[f(x)]+\lim_{x\to a}[g(x)]$

2. Difference Rule: $\lim_{x\to a}[f(x)-g(x)] = L-M = \lim_{x\to a}[f(x)]-\lim_{x\to a}[g(x)]$

3. Product Rule: $\lim_{x\to a}[f(x)g(x)] = LM = \lim_{x\to a}[f(x)]\lim_{x\to a}[g(x)]$

4. Constant Multiple Rule: $\lim_{x\to a} kf(x) = kL = k\lim_{x\to a} f(x), \quad k \in R$.

5. Quotient Rule: $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad M \neq 0$.

6. Power Rule: If m and n are integers, then $\lim_{x\to a}[f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$, provided $L^{\frac{m}{n}}$ is finite.

7. Identity Function: If f is an identity function f(x) = x then for any value of a:

$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a.$$

8. Constant Function: If f is a constant function, that is; f(x) = k then for any value of a:

$$\lim_{x\to a} f(x) = \lim_{x\to a} k = k.$$

Theorem 2: If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to a} P(x) = P(a) = a_n a^n + a_{n-1}a^{n-1} + \cdots + a_0.$$

Theorem 3: If P(x) and Q(x) are polynomials and $Q(x) \neq 0$, then

$$\lim_{x\to a}[P(x)/Q(x)] = P(a)/Q(a)$$

For example, $\lim_{x\to 1}\dfrac{x^3+4x^2-3}{x^2+5} = \dfrac{1+4-3}{1+5} = \dfrac{2}{6} = \dfrac{1}{3}$.

Theorem 5 applies only when the denominator of the rational function is not zero at the limiting point `a`. If the denominator is 0, canceling common factors in the numerator and denominator will sometimes reduce the fraction to one whose denominator is no longer zero at `a`. When this happens, we can find the limit by substitution in the simplified fraction.

REMARK: It may be noted that almost every problem of limit involves indeterminate expression such as (0/0), (∞ /∞), (0 × ∞), (∞ - ∞), (0^0), (1^∞), (∞^0). When such expressions occur we have to workout the given problem to get some definite value. We shall show you this in coming examples and theorems.

Example 03: (Canceling a common factor) Evaluate $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

Solution: We cannot just substitute x = 1, because it makes the denominator zero. However, we can factorize the numerator and denominator and cancel the common factor to obtain:

$$\frac{x^2+x-2}{x^2-x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x}, \quad x \neq 1. \text{ Thus, } \lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{x+2}{x} = \frac{1+2}{1} = 3$$

Example 04: Evaluate $\lim_{x\to 2}\dfrac{x^3-8}{x^2-4}$

$$=1^1.1^1.\frac{25}{81}=\frac{25}{81}$$

2. Prove that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Proof: Using binomial theorem, we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left\{1+x\frac{1}{x}+\frac{x(x-1)}{2!}\cdot\frac{1}{x^2}+\frac{x(x-1)(x-2)}{3!}\cdot\frac{1}{x^3}+\dots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{x^2(1-1/x)}{2!x^2}+\frac{x^3(1-1/x)(1-2/x)}{3!x^3}+\dots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{(1-1/x)}{2!}+\frac{(1-1/x)(1-2/x)}{3!}+\dots\right\}$$

Applying the limit on the right hand side, we get

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots \qquad \left[\text{Note: } \lim_{x\to\infty}\frac{c}{x}\to 0\right]$$

The sum of this well-known infinite series is approximately equal to 2.71828. This sum always lies between 2 and 3 and is denoted by e. Thus:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots=e$$

Corollary: In the above limit, if we put y = 1/x then y will approach to 0 as x approaches to ∞. Thus, $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim\limits_{y\to 0}(1+y)^{1/y}=e$

REMARK: It may be noted that $1-1+\frac{1}{2!}-\frac{1}{3!}+\dots=e^{-1}=\frac{1}{e}$

Example 08: Evaluate (i) $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x$ **(ii)** $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{mx}$ **(iii)** $\lim\limits_{x\to\infty}\left(1+\frac{m}{x}\right)^x$

Solution: $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\lim\limits_{x\to\infty}\left[1+\left(-\frac{1}{x}\right)\right]^x=\lim\limits_{x\to\infty}\left\{\left[1+\left(-\frac{1}{x}\right)\right]^{-x}\right\}^{-1}=(e)^{-1}=\frac{1}{e}$

(ii) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{mx}=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^m=(e)^m=e^m$

(iii) $\lim\limits_{x\to\infty}\left(1+\frac{m}{x}\right)^x=\lim\limits_{x\to\infty}\left[\left(1+\frac{m}{x}\right)^{x/m}\right]^m=(e)^m=e^m$

3. Prove that $\boxed{\lim\limits_{x\to\infty}\frac{a^x-1}{x}=\ln a}$ → N.B

Solution: If we place x = 0 in the above function, we get once 0/0 which is an indeterminate expression. To find the required limit, we proceed as under:

Putting $a^x-1=y$ ➔ $a^x=1+y$. Taking log on each side, we get

Moreover, as x tends to 0, y also tends to 0. Hence,

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\frac{\ln(1+y)}{\ln a}}=\lim_{y\to 0}\frac{\ln a}{\frac{\ln(1+y)}{y}}=\lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln(1+y)}$$

$$=\lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}}=\ln a\left\{\lim_{y\to 0}\frac{1}{\ln(1+y)^{1/y}}\right\}=\ln a\left(\frac{1}{\ln e}\right)=\ln a$$

➔ $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$ **Note:** $\lim_{y\to 0}(1+y)^{1/y}=e$, $\ln e=1$.

Example 09: Evaluate the following limits:

(i) $\lim_{x\to 0}\frac{25^x-1}{16^x-1}$ **(ii)** $\lim_{x\to 0}\frac{25^x-16^x}{x}$ **(iii)** $\lim_{x\to 0}(1+3x)^{1/x}$

Solution: (i) $\lim_{x\to 0}\frac{25^x-1}{16^x-1}=\lim_{x\to 0}\left(\frac{25^x-1}{x}\times\frac{x}{16^x-1}\right)=\lim_{x\to 0}\frac{25^x-1}{x}\times\lim_{x\to 0}\frac{x}{16^x-1}$

$$=\frac{\log 25}{\log 16}=\frac{\log 5^2}{\log 4^2}=\frac{2\log 5}{2\log 4}=\frac{\log 5}{\log 4}\quad\Rightarrow\quad\lim_{x\to 0}\frac{25^x-1}{16^x-1}=\frac{\log 5}{\log 4}$$

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(ii) $\lim_{x\to 0}\frac{25^x-16^x}{x}=\lim_{x\to 0}\frac{25^x-1-16^x+1}{x}=\lim_{x\to 0}\frac{(25^x-1)-(16^x-1)}{x}$

$$=\lim_{x\to 0}\left\{\frac{(25^x-1)}{x}-\frac{(16^x-1)}{x}\right\}=\lim_{x\to 0}\frac{(25^x-1)}{x}-\lim_{x\to 0}\frac{(16^x-1)}{x}$$

Thus, $\lim_{x\to 0}\frac{25^x-16^x}{x}=\log 25-\log 16=\log\left(\frac{25}{16}\right)=\log\left(\frac{5}{4}\right)^2=2\log\left(\frac{5}{4}\right)$

(iii) $\lim_{x\to 0}(1+3x)^{1/x}=\lim_{x\to 0}\left[(1+3x)^{1/3x}\right]^3=[e]^3=e^3$

Example 10: Evaluate (i) $\lim_{x\to 0}\frac{\ln(x+h)-\ln h}{x}$ (ii) $\lim_{x\to 0}\frac{\sin(x+h)-\sin h}{x}$

Solution: (i)

$$\lim_{x\to 0}\frac{\log(x+h)-\ln h}{x}=\lim_{x\to 0}\frac{1}{x}\{\log(x+h)-\log h\}=\lim_{x\to 0}\frac{1}{x}\log\left(\frac{x+h}{h}\right)=\lim_{x\to 0}\frac{1}{x}\log\left(1+\frac{x}{h}\right)$$

$$=\lim_{x\to 0}\log\left(1+\frac{x}{h}\right)^{1/x}=\lim_{x\to 0}\left\{\log\left(1+\frac{x}{h}\right)^{h/x}\right\}^{1/h}=\log\left\{\lim_{x\to 0}\left(1+\frac{x}{h}\right)^{h/x}\right\}^{1/h}$$

$\lim_{x\to 0}\frac{\log(x+h)-\log h}{x}=\log e^{1/h}=\frac{1}{h}\log e=\frac{1}{h}$. since, $\lim_{x\to 0}(1+x)^{1/x}=e$ & and $\log e=1$ (

ii) Since, $\sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$.

$$\lim_{x\to 0}\frac{\sin(x+h)-\sin h}{x}=\lim_{x\to 0}\frac{2\cos\left(\frac{x+h+h}{2}\right)\sin\left(\frac{x+h-h}{2}\right)}{x}$$

$$= \lim_{x\to 0}\frac{\cos\left(\frac{x+2h}{2}\right)\sin\left(\frac{x}{2}\right)}{(x/2)} = \lim_{x\to 0}\frac{\sin x/2}{x/2}\lim_{x\to 0}\cos\left(\frac{x+2h}{2}\right) = 1.\cos\left(\frac{2h}{2}\right) = \cos h$$

Thus, $\lim_{x\to 0}\frac{\sin(x+h)-\sin h}{x} = \cos h$

Example 11: Evaluate the following limits:

(i) $\lim_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{1-x^3}\right] = \lim_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{(1-x)(1+x+x^2)}\right] = \lim_{x\to 1}\left[\frac{1+x+x^2-3}{(1-x)(1+x+x^2)}\right]$

$$= \lim_{x\to 1}\left[\frac{x^2+x-2}{(1-x)(1+x+x^2)}\right] = \lim_{x\to 1}\left[\frac{(x+2)(x-1)}{(1-x)(1+x+x^2)}\right] = \lim_{x\to 1}\left[\frac{(x+2)(x-1)}{-(x-1)(1+x+x^2)}\right]$$

$$= \lim_{x\to 1}\left[\frac{(x+2)}{-(1+x+x^2)}\right] = \frac{3}{-3} = -1$$

(ii) $\lim_{x\to 0}\left[\frac{\operatorname{cosec} x-\cot x}{x}\right] = \lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right]\times\frac{1}{x} = \lim_{x\to 0}\left[\frac{1-\cos x}{x\sin x}\right]$

$$= \lim_{x\to 0}\left[\frac{1-\cos x}{x\sin x}\right]\times\left[\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\left[\frac{1-\cos^2 x}{x\sin x(1+\cos x)}\right] = \lim_{x\to 0}\left[\frac{\sin^2 x}{x\sin x(1+\cos x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x}{x}\right]\times\lim_{x\to 0}\left[\frac{1}{(1+\cos x)}\right] = 1\times\frac{1}{2} = 1/2$$

(iii) $\lim_{y\to x}\frac{y^{2/3}-x^{2/3}}{y-x}$. Putting $y = x+h \Rightarrow h = x-y$. As $y\to x$ then $h\to 0$. Thus,

$\lim_{y\to x}\frac{y^{2/3}-x^{2/3}}{y-x} = \lim_{h\to 0}\frac{[x+h]^{2/3}-x^{2/3}}{x+h-x}$. Apply binomial expansion, we get :

$$\lim_{h\to 0}\frac{x^{2/3}+\frac{2}{3}x^{\frac{2}{3}-1}h+\frac{2}{3}\left(\frac{2}{3}-1\right)x^{\frac{2}{3}-2}h^2+\ldots-x^{2/3}}{h} = \lim_{h\to 0}\frac{\frac{2}{3}h\left[x^{\frac{2}{3}-1}+\left(\frac{2}{3}-1\right)x^{\frac{2}{3}-2}h+\ldots\right]}{h}$$

$$= \lim_{h\to 0}\frac{2}{3}\left[x^{\frac{2}{3}-1}+\left(\frac{2}{3}-1\right)x^{\frac{2}{3}-2}h+\ldots\right] = \frac{2}{3}x^{-1/3}$$

(iv) $\lim_{x\to\pi}\frac{\tan(\sin x)}{\sin x}$. Putting $\sin x = z$. As $x\to\pi$ then $z\to\sin\pi = 0$. Thus,

$$\lim_{x\to\pi}\frac{\tan(\sin x)}{\sin x} = \lim_{z\to 0}\frac{\tan z}{z} = \lim_{z\to 0}\frac{\sin z}{z.\cos z} = \lim_{z\to 0}\frac{\sin z}{z}\times\lim_{z\to 0}\cos z = 1.1 = 1$$

(v) $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0}x\times\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = \lim_{x\to 0}x\times A = 0.A = 0$

[Supposing that $\lim_{x\to 0}\sin(1/x) = A$]

REMARK: The limit $\lim_{x\to 0}\sin(1/x)$ is discussed in the next session.

Limit at Infinity

Sometimes we are concerned with the behavior of a function f(x) as the magnitude of the variable say x, increases without bound (that is; the magnitude becomes infinitely large). The limits studied in such instances are called limits at infinity and are written as:

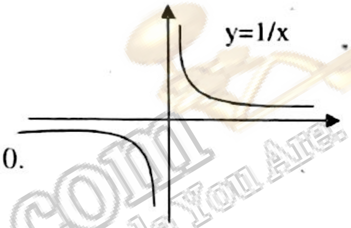
$\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$

The notation $x \to \infty$ can be read, "as x increases without bound" or "as x tends toward infinity." Similarly, $x \to -\infty$ is read, "as x decreases without bound" or "as x tends toward minus infinity."

REMARK: The English mathematician John Wallis (1616 – 1703) was the first to use the symbol ∞ for infinity.

For instance, consider the function f(x) = 1/x. Larger x becomes; closer 1/x gets to zero. The table that follows illustrates this statement. The graph of f(x) = 1/x also shows that as x becomes larger, 1/x gets closer to zero.

x	±2	±10	±100	±1000
y=1/x	±0.5	±0.1	±0.01	±0.001



We say that the limit of 1/x as x increases without bound is 0. This is written as $\lim\limits_{x\to\infty} 1/x = 0$.

In a similar manner, we can determine that $\lim\limits_{x\to-\infty} 1/x = 0$.

The above limits merely describe the behavior of 1/x as the magnitude of x increases without bound. They describe the tendency of 1/x toward zero as x tends toward infinity. Consider next the function defined by

$f(x) = 1/x^n$, n is a positive integer.

Comparing 1/x and $1/x^n$, noting that the magnitude of x^n is larger than the magnitude of x when x is approaching infinity or minus infinity. It then follows that:

$\lim\limits_{x\to\infty} 1/x^n = 0$ and $\lim\limits_{x\to-\infty} 1/x^n = 0, \ (n \in Z^+)$.

If the function is $f(x) = c/x^n$ where c is a finite constant, then the limit as x increases or decreases without bound will still be zero. Thus: $\lim\limits_{x\to\pm\infty} c/x^n$,

Example 12: Evaluate the following limits

(i) $\lim\limits_{x\to\infty} \dfrac{3x^3 - 8x + 1}{4x^3 - 3x^2 - 16}$

Solution: We notice that if the value of x is placed ∞ directly, we get an expression of the form ∞/∞. In order to solve this problem, we notice that the functions in the numerator and denominator are polynomials and the limit is taken at infinity. In such cases, one should take common, the term with highest power of x, both from the numerator and denominator. Doing this, we get

$$\lim_{x\to\infty} \frac{3x^3 - 8x + 1}{4x^3 - 3x^2 - 16} = \lim_{x\to\infty} \frac{x^3\left(3 - 8/x^2 + 1/x^3\right)}{x^3\left(4 - 3/x - 16/x^3\right)} = \lim_{x\to\infty} \frac{\left(3 - 8/x^2 + 1/x^3\right)}{\left(4 - 3/x - 16/x^3\right)}$$

Applying the limit now and using the result shown in the above box, we get

$$\lim_{x\to\infty} \frac{3x^3 - 8x + 1}{4x^3 - 3x^2 - 16} = \frac{3}{4}$$

(ii) $\lim_{x\to\infty}\left(x^3+2x^2+1\right)$

Solution: Here we see that there is an algebraic expression in the numerator only. To solve such limit problem, we substitute x = 1/y so that y = 1/x Now if x tends to infinity, y will tend to zero.

$$\lim_{x\to\infty}\left(x^3+2x^2+1\right)=\lim_{y\to 0}\left(\frac{1}{y^3}-2\frac{1}{y^2}+1\right)=\lim_{y\to 0}\left(\frac{1-2y+y^3}{y^3}\right)=\frac{1}{0}=\infty.$$

This shows that the above limit does not exist.

REMARK: The idea of dividing each term by a power of *x* is used only for limits at infinity. It will not help in the evaluation of other kinds of limits.

(iii) $\lim_{x\to\infty}\frac{x^4+5x-6}{x^3+x^2-7}=\lim_{x\to\infty}\frac{x^4\left(1+5/x^3-6/x^4\right)}{x^3\left(1+1/x-6/x^3\right)}=\lim_{x\to\infty}\frac{x\left(1+5/x^3-6/x^4\right)}{\left(1+1/x-6/x^3\right)}$

$=\infty\frac{1+0-0}{1+1-0}=\infty$. Hence, limit of given function does not exist.

(iv) $\lim_{x\to\infty}\frac{x^2+5}{x^{3/2}+7}=\lim_{x\to\infty}\frac{x^2\left(1+5/x^2\right)}{x^{3/2}\left(1+7/x^{3/2}\right)}=\lim_{x\to\infty}\frac{x^{1/2}\left(1+5/x^2\right)}{\left(1+7/x^{3/2}\right)}=\infty\frac{1+0}{1+0}=\infty$

Thus, limit of given function does not exist.

(v) $\lim_{x\to\infty}\frac{a^x-1}{x}$, $a>0$. Putting $a=1+h$, we get

$$\lim_{x\to\infty}\frac{x\left[h+x(x-1)h^2/2!+...\right]}{x}=\lim_{x\to\infty}\left[h+x(x-1)h^2/2!+...\right]=h+\infty+\infty+..=\infty$$

$$\lim_{x\to\infty}\frac{a^x-1}{x}=\lim_{x\to\infty}\frac{(1+h)^x-1}{x}=\lim_{x\to\infty}\frac{1+xh+x(x-1)h^2/2!+...-1}{x}$$

Thus limit of given function does not exist.

(vi) $\lim_{x\to\infty}\left[\frac{x^2}{x+1}-\frac{x^2}{x+3}\right]=\lim_{x\to\infty}x^2\left[\frac{1}{x+1}-\frac{1}{x+3}\right]=\lim_{x\to\infty}x^2\left[\frac{x+3-x-1}{(x+1)(x+3)}\right]$

$=\lim_{x\to\infty}x^2\left[\frac{2}{(x^2+4x+3)}\right]=\lim_{x\to\infty}\frac{x^2}{x^2}\left[\frac{2}{(1+4/x+3/x^2)}\right]=\frac{2}{1+0+0}=2$

(vii) $\lim_{x\to\infty}\frac{\sqrt{x^2+4}}{x-6}=\lim_{x\to\infty}\frac{\sqrt{x^2\left(1+4/x^2\right)}}{x(1-6/x)}=\lim_{x\to\infty}\frac{x\sqrt{\left(1+4/x^2\right)}}{x(1-6/x)}$

$=\lim_{x\to\infty}\frac{\sqrt{\left(1+4/x^2\right)}}{(1-6/x)}=\frac{\sqrt{1+0}}{1-0}=1$

(viii) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x=\lim_{x\to 0}\left(\frac{x+1}{x}\right)^{-x}=\lim_{x\to 0}\left(\frac{x}{x}+\frac{1}{x}\right)^{-x}=\lim_{x\to 0}\left[\left(1+\frac{1}{x}\right)^x\right]^{-1}=e^{-1}$

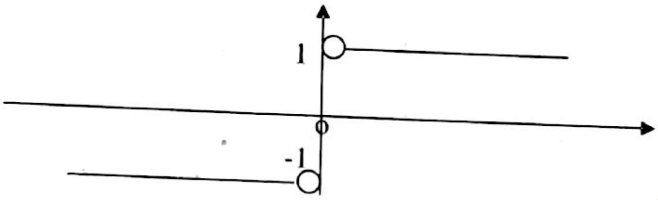
(ix) $\lim_{x\to\infty}\frac{x+\sin x}{x}=\lim_{x\to\infty}\left[\frac{x}{x}+\frac{\sin x}{x}\right]=\lim_{x\to\infty}\left[1+\frac{\sin x}{x}\right]=1+\lim_{x\to\infty}\frac{\sin x}{x}=1+0=1$

REMARK: $|\sin x|\le 1$. This sin x/x always approaches zero as x tends to infinity.

One Sided Limits (Left Hand and Right Hand Limits)

To have a limit L as x approaches 'a', a function f must be defined on both sides of 'a', and its values f(x) must approach L as x approaches 'a' from either side. Because of this, ordinary limits are sometimes called two–sided limits.

It is possible for a function to approach a limiting value as x approaches 'a' from only one side, either from the right or from the left. In this case we say that f has a one–sided (either right hand or left hand) limit at 'a'. For instance, the function $f(x) = x/|x|$ graphed below has limit 1 as x approaches zero from the right, and limit –1 as x tends zero from the left.



Let f(x) be defined on an interval (a, b) where a < b. If f(x) approaches arbitrarily close to L as x approaches 'a' from right within the interval (a, b), then we say that f has right-hand limit L_1 at 'a' and we write: $\lim\limits_{x \to a^+} f(x) = L_1$

Similarly, let f(x) be defined on an interval (a, b) where a < b. If f(x) approaches arbitrarily close to L_2 as x approaches 'a' from left within the interval (a, b), then we say that f has left–hand limit L_2 at 'a', and we write: $\lim\limits_{x \to a^-} f(x) = L_2$

For the function $f(x) = \dfrac{x}{|x|}$, we have, $\lim\limits_{x \to 0^+} f(x) = 1$ and $\lim\limits_{x \to 0^-} f(x) = -1$.

REMARKS:

(i) The left hand limit of a function f(x) is shown in two different ways.

$$f(a-0) \text{ or } \lim_{x \to a-0} f(x)$$

(ii) Similarly, the right hand limit may be shown as:

$$f(a+0) \text{ or } \lim_{x \to a+0} f(x)$$

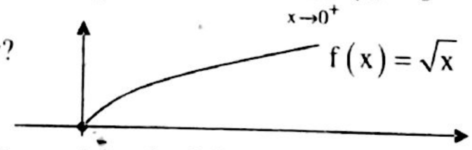
(iii) If f(a – 0) = f(a + 0) = L; then $\lim\limits_{x \to a} f(x)$ exits and is equal to L. This means that if the right and left limits of a function exit and are equal to L then $\lim\limits_{x \to a-0} f(x) = L$.

(iv) If left and right limits of a function f(x) are not equal, we say that limit of f(x) does not exist.

Example 13: Determine $\lim\limits_{x \to 0} \sqrt{x}$

Solution: At first glance this limit may look simple, since you could easily believe that the limit is zero. However, the limit is not zero. Although it is true that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$.

It is also true that $\lim\limits_{x \to 0^-} \sqrt{x}$ does not exist. Why?



A function defined only for x ≥ 0

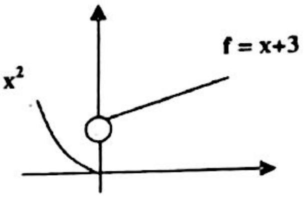
Because if x approaches from the left means x is a negative number, and $\sqrt{x}$ is not defined for such numbers. It may be noted that a function must be defined for all values of x or else the limit does not exist. The following figure shows the graph of $f(x)=\sqrt{x}$, which is defined only for $x \geq 0$.

Our Conclusion:

$\lim\limits_{x\to 0^+} \sqrt{x} = 0$ and $\lim\limits_{x\to 0^-} \sqrt{x}$ does not exist. Hence, $\lim\limits_{x\to 0} \sqrt{x}$ does not exist.

Example 13: Determine $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^-} f(x)$ for the function defined by

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x+3 & x > 0 \end{cases}$$



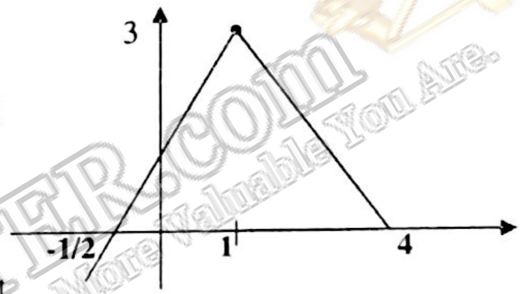
Solution: In this piecewise function, the value of f(x) is computed as x^2 when $x \leq 0$ and as x + 3 when x > 0. Thus,

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (x+3) = 3$ and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x^2) = 0$.

Since the limits are different, we conclude that $\lim\limits_{x\to 0} f(x)$ does not exist.

Example 14: Determine $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^-} f(x)$ for the function defined by

$$f(x) = \begin{cases} 2x+1 & x \leq 1 \\ 4-x & x > 0 \end{cases}$$



Solution: Here $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (4-x) = 3$ and

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2x+1) = 3$.

Since the right – hand and left–hand limits are same, that is; 3, therefore, we conclude that $\lim\limits_{x\to 1} f(x) = 3$.

Example 16: Evaluate $\lim\limits_{x\to 2-0} \sqrt{\dfrac{4-x^2}{6-5x+x^2}}$

Solution: We observe that the given function is one valued function. To evaluate the limit of such function, we put x = 2 – h. This gives h = 2 – x. Now as x tends to 2, h will tend to zero. Thus,

$$\lim_{x\to 2-0} \sqrt{\frac{4-x^2}{6-5x+x^2}} = \lim_{h\to 0} \sqrt{\frac{4-(2-h)^2}{6-5(2-h)+(2-h)^2}} = \lim_{h\to 0} \sqrt{\frac{4-4+4h-h^2}{6-10+5h+4-4h+h^2}}$$

$$= \lim_{h\to 0} \sqrt{\frac{4h-h^2}{h+h^2}} = \lim_{h\to 0} \sqrt{\frac{h(4-h)}{h(1+h)}} = \lim_{h\to 0} \sqrt{\frac{(4-h)}{(1+h)}} = \sqrt{\frac{(4-0)}{(1+0)}} = \sqrt{\frac{4}{1}} = 2$$

Example 17: If $f(x) = \left(\dfrac{x^2-4}{x-2}\right)$ **find f(2 – 0) and f(2 + 0). Hence find** $\lim\limits_{x\to 2} f(x)$.

Solution: By definition, $f(2-0) = \lim\limits_{x\to 2-0} f(x) = \lim\limits_{x\to 2-0} \dfrac{(x^2-4)}{(x-2)}$.

Put x = 2 – h ➔ h = 2 – x. As x ➔ 2, h ➔ 0. Thus,

$$f(2-0) = \lim_{x\to 2-0} f(x) = \lim_{x\to 2-0} \frac{(x^2-4)}{(x-2)} = \lim_{h\to 0} \frac{(2-x)^2-4}{(2-h)-2} = \lim_{h\to 0} \frac{4-4h+h^2-4}{2-h-2}$$

$$= \lim_{h\to 0} \frac{h(h-4)}{-h} = \lim_{h\to 0} \frac{h-4}{-1} = 4$$

Similarly, $f(2+0) = \displaystyle\text{Lim}_{x\to 2+0} f(x) = \text{Lim}_{x\to 2+0} \frac{(x^2-4)}{(x-2)}$.

We put x = 2 + h ➔ h = x - 2. As x ➔ 2, h ➔ 0. Thus,

$$f(2+0) = \lim_{x\to 2+0} f(x) = \lim_{x\to 2+0} \frac{(x^2-4)}{(x-2)} = \lim_{h\to 0} \frac{(2+x)^2-4}{(2+h)-2} = \lim_{h\to 0} \frac{4+4h+h^2-4}{2+h-2}$$

$$= \lim_{h\to 0} \frac{h(h+4)}{h} = \lim_{h\to 0} \frac{h+4}{1} = 4$$

Since f(2 – 0) = f(2 + 0) = 4, we deduce that $\lim_{x\to 2} f(x) = 4$.

Example 18: Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 3-0}\left[\frac{1}{x-3} - \frac{1}{|x-3|}\right]$. Putting x = 3 – h. As x tends to 3 then h tends to 0.

Thus, $\displaystyle\lim_{x\to 3-0}\left[\frac{1}{x-3} - \frac{1}{|x-3|}\right] = \lim_{h\to 0}\left[\frac{1}{3-h-3} - \frac{1}{|3-h-3|}\right] = \lim_{h\to 0}\left[\frac{1}{-h} - \frac{1}{|-h|}\right]$

$\displaystyle= \lim_{h\to 0}\left[-\frac{1}{h} - \frac{1}{h}\right] = \lim_{h\to 0} \frac{-2}{h} = -\infty$. Thus limit of given function does not exist.

(b) $\displaystyle\lim_{x\to 2-0} \frac{x^2+2x-8}{x^2-4}$. Putting x = 2 – h. As x tends 2 then h tends to 0. Thus,

$$\lim_{x\to 2-0} \frac{x^2+2x-8}{x^2-4} = \lim_{h\to 0} \frac{(2-h)^2+2(2-h)-8}{(2-h)^2-4} = \lim_{h\to 0} \frac{4-4h+h^2+4-2h-8}{4-4h+h^2-4}$$

$$= \lim_{h\to 0} \frac{h^2-6h}{h^2-4h} = \lim_{h\to 0} \frac{h(h-6)}{h(h-4)} = \lim_{h\to 0} \frac{(h-6)}{(h-4)} = \frac{3}{2}$$

(c) $\displaystyle\lim_{x\to 1^-} \frac{\sqrt{1-x^2}}{1-x} = \lim_{x\to 1^-} \frac{\sqrt{(1-x)(1+x)}}{\sqrt{(1-x)}\sqrt{(1-x)}} = \lim_{x\to 1^-} \frac{\sqrt{(1-x)}\sqrt{(1+x)}}{\sqrt{(1-x)}\sqrt{(1-x)}} = \lim_{x\to 1^-} \frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{\sqrt{2}}{0} = \infty$

Alternatively putting x = 1 – h. As x tends to 1 then h tends to 0. Thus,

$$\lim_{x\to 1^-} \frac{\sqrt{1-x^2}}{1-x} = \lim_{h\to 0} \frac{\sqrt{1-(1-h)^2}}{1-(1-h)} = \lim_{h\to 0} \frac{\sqrt{1-(1-2h+h^2)}}{1-1+h} = \lim_{h\to 0} \frac{\sqrt{2h-h^2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{h}\sqrt{2-h}}{\sqrt{h}\sqrt{h}} = \lim_{h\to 0} \frac{\sqrt{2-h}}{\sqrt{h}} = \sqrt{\frac{2}{0}} = \infty$$

(iv) $\displaystyle\lim_{x\to 0} \frac{|-1-x|-1}{|x|}$.

Consider f(0 – 0). Put x = 0 – h ➔ h = - x. As x ➔ 0 then h ➔ 0. Thus

$$f(0-0) = \lim_{h\to 0} \frac{|-1+h|-1}{|-h|} = \lim_{h\to 0} \frac{|-(1-h)|-1}{|-h|} = \lim_{h\to 0} \frac{(1-h)-1}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

Now consider f(0 + 0). Put x = 0 + h ➔ h = x. As x ➔ 0 then h ➔ 0. Thus

$$f(0+0) = \lim_{h\to 0} \frac{|-1-h|-1}{|h|} = \lim_{h\to 0} \frac{|-(1+h)|-1}{|h|} = \lim_{h\to 0} \frac{(1+h)-1}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Since f (0 - 0) ≠ f (0 + 0), hence $\lim_{x\to 0} f(x)$ does not exist.

(v) $\lim_{x\to 0-0} \frac{x}{x-|x|}$.

Putting x = 0 – h = -h. As x tends to 0 then h also tends to 0. Thus

$$\lim_{x\to 0-0} \frac{x}{x-|x|} = \lim_{h\to 0} \frac{-h}{-h-|-h|} = \lim_{h\to 0} \frac{-h}{-h-h} = \lim_{h\to 0} \frac{-h}{-2h} = -\frac{1}{2}.$$

(vi) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Consider $f(0-0) = \lim_{x\to 0-0} \sin\left(\frac{1}{x}\right)$. Putting x = 0 – h. As x tends to zero then x tends to zero. Thus,

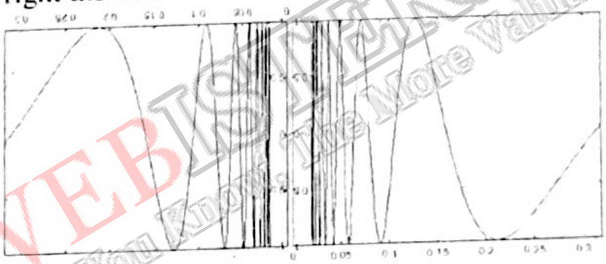
$$f(0-0) = \lim_{h\to 0} \sin\left(\frac{-1}{h}\right) = -\lim_{h\to 0} \sin\left(\frac{1}{h}\right) = -A \text{ (say)}.$$

Similarly, putting x = 0 + h. As x tends to zero then x tends to zero. Thus,

$$f(0+0) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right) = A$$

Since, f (0 – 0) and f (0 + 0) are not equal. Thus, $\lim_{x\to 0} \sin(1/x)$ does not exist.

The graph of this function is shown below. Readers may observe that as x approaches zero from left and right the values of sin (1/x) are not same.



Left and Right Limits of Piecewise Functions

In this section we shall discuss the limits of piecewise functions. It may be noted that we have already mentioned that if left limit f(a - 0) and right limit f(a + 0) are equal L where L is a finite number then limit of function f(x) when x approaches `a` is also equal to L.

Example 19: Do as directed.

(i) Find $\lim_{x\to 2} f(x)$ where f(x) is defined as: $f(x) = \begin{cases} x^2 - 1 & x \le 2 \\ \sqrt{x+7} & x > 2 \end{cases}$

Solution: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^2-1) = 3$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2}\left(\sqrt{x+7}\right) = 3$.

Thus, $\lim_{x\to 2} f(x) = 3$, since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$

Explanation: Given function is piecewise function. One of its part is defined for $x \le 2$ and the other one.is for x > 2. While evaluating the left limit at x = 2, we take that part of function which is on the left of 2. Similarly, when valuating the right limit at x = 2, we take that part of function which is on the right of 2

(ii) Find $\lim_{x\to 1} f(x)$ where f(x) is defined as: $f(x) = \begin{cases} x^2 & x \le 1 \\ x^3 & x > 1 \end{cases}$

Solution: Since function is piecewise function, hence to evaluate the required limit, we shall consider left and right hand limits at x = 1.

Now, $\lim_{x\to 1-0} f(x) = \lim_{x\to 1}(x^2) = 1$. Also $\lim_{x\to 1+0} f(x) = \lim_{x\to 1}(x^3) = 1$ Since both left and right limits at x = 1 are equal, we conclude that $\lim_{x\to 1} f(x) = 1$

(iii) Find 'a' if $\lim_{x\to -1} f(x)$ exists where, $f(x)$ is defined as : $f(x) = \begin{cases} 2x+1 & x \le -1 \\ ax^2 & x > -1 \end{cases}$

Solution: Since function is piecewise function, hence to evaluate the required limit, we shall consider left and right hand limits at x = -1.

Now, $\lim_{x\to -1-0} f(x) = \lim_{x\to -1}(x+2) = 1$. Also $\lim_{x\to -1+0} f(x) = \lim_{x\to -1}(ax^2) = a$

Since $\lim_{x\to 1} f(x)$ exists hence, both left and right limits at x = -1 are equal.

This gives a = 1.

(vi) Find 'a' if $\lim_{x\to 0} f(x)$ exists where, $f(x)$ is defined as : $f(x) = \begin{cases} \cos x & x \le 0 \\ a + x & x > 0 \end{cases}$

Solution: Since function is two-valued function, hence to evaluate the required limit, we shall consider left and right hand limits at x = -1.

Now, $\lim_{x\to 0-0} f(x) = \lim_{x\to 0} \cos x = 1$. Also $\lim_{x\to 0+0} f(x) = \lim_{x\to 0}(a+x) = a$

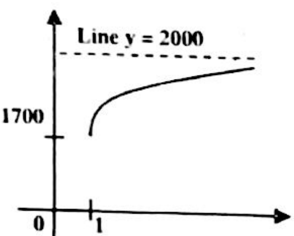
Since $\lim_{x\to 1} f(x)$ exists hence, both left and right limits at x = 0 are equal. This gives a = 1.

Applications of Limits

Example 19: Suppose that profit from the sale of x units of microchips is $P(x) = 2000 - \frac{300}{x}$ dollars for $x \ge 1$. What will be the maximum profit if the sale increases indefinitely?

Solution: Clearly, we are asked to find the limit of P(x) as x tends to infinity. Now

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty}\left(2000 - \frac{300}{x}\right) = 2000 \cdot \left[\text{Note: } \lim_{x\to\infty}\frac{300}{x} = 0\right]$$



Interpretation:

No matter how big sale of microchips is made, the profit will never exceed $2000. The graph of above function is depicted in the adjacent figure.

Example 20: The height of tree grows according to law h(t) = (9.5t – 2)/(t + 1) where h is in feet and t is the time in years. Find the height of the tree after (a) one year (b) 2 years (c) 10 years and (d) 30 years. What will be maximum height of the tree as the time passes indefinitely?

Solution: (a) Put t = 1, we get: h = 7.5/3 = 2.5 feet

(b) Put t = 2, we get: h = 17/3 = 5.67 feet

(c) Put t = 10, we get: h = 93/11 = 8.45

(d) Put t = 30, we get : h = 9.2

(e) If t increases indefinitely, that as t tends to ∞, we get

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty}\left(\frac{9.5t-2}{t+1}\right) = \lim_{t\to\infty}\left(\frac{t(9.5t-2/t)}{t(1+1/t)}\right) = \left(\frac{9.5-0}{1+0}\right) = 9.5$$

This means that maximum height of the tree will reach 9.5 feet.

Example 21: The number of degrees `d` in each interior angle of a regular polygon of 'n' sides is d = (180n – 360)/n. Find the measure of interior angle of regular hexagon. Find the limit, that is; the number of degree toward which the angles tend as n gets larger and larger.

Solution: We have solved this problem in chapter one where we showed that for an equilateral triangle above formula gives $d = 60^o$, for square (n = 4), $d = 90^o$ and for n = 5 (Regular pentagon), $d = 108^o$ etc. Putting n = 6, we get

$$d = [180(6) - 360]/6 = 120^o$$

Now as n gets larger and larger means n tends towards infinity, that is; if `n` gets larger and larger, we get:

$$\lim_{n\to\infty} d = \lim_{n\to\infty} \frac{180n - 360}{n} = \lim_{n\to\infty} \frac{n(180 - 360/n)}{n} = \lim_{n\to\infty}(180 - 0) = 180$$

This shows that the number of degrees tend towards 180^o as n gets larger and larger.

Example 22: A woman with a temperature of 103^o F is given medicine that will reduce her temperature. The medicine takes one hour before it begins to work, and after that (that is, for $t \geq 1$) her temperature at t hours will be

$$T(t) = 103.7 - (5.1t - 4.5)/t \text{ degrees}$$

(a) What is her temperature at t = 1 hour, t = 3 hours and t = 10 hours?

(b) Eventually, to what temperature is her body reduced?

Solution: (a) The temperature of the woman at t = 1 hour will be

$$T(1) = 103.7 - \frac{5.1(1) - 4.5}{(1)} = 103.7 - 0.6 = 103.1 \text{ degrees}$$

The temperature at t = 3 hours will be

$$T(3) = 103.7 - \frac{5.1(3) - 4.5}{(3)} = 103.7 - 3.6 = 100.1 \text{ degrees}$$

Finally, the temperature at t = 10 hours will be

$$T(10) = 103.7 - \frac{5.1(10) - 4.5}{(10)} = 103.7 - 4.65 = 99.05 \text{ degrees}$$

(b) The temperature of the woman will be reduced if t gets larger and larger, that is; $t \to \infty$. Thus, $\lim_{t\to\infty} T(t) = \lim_{t\to\infty}\left(103.7 - \frac{5.1t - 4.5}{t}\right) = 103.7 - \lim_{t\to\infty}\frac{t\left(5.1 - \frac{4.5}{t}\right)}{t}$

$$= 103.7 - 5.1 = 98.6 \text{ degrees.} \quad \left[\text{using } \lim_{n\to\infty}\frac{c}{x} \to 0\right]$$

This shows that as time is prolonged the temperature of the woman will reduce to 98.6^o F.

2.2 CONTINUOUS AND DISCONTINUOUS FUNCTIONS

In this section we shall discuss a very important concept of calculus known as "Continuity".

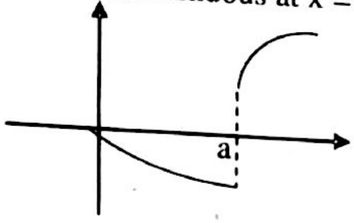
Definition: A function f(x) is said to be continuous at a point x = a if any one of the following two conditions hold.

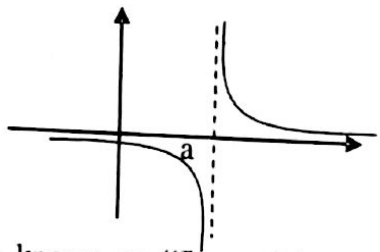
(i) $f(a - 0) = f(a + 0) = f(a) = L$, where f(a) is value of function at x = a.

(ii) $\lim_{x\to a} f(x) = f(a) = L$, L being a finite number.

REMARK: (i) Some authors are of the opinion that if left and right limits are equal and finite then function is continuous. This is not true unless they both are equal to f(a).

However, if it is given that function f?(x) is continuous at x = a then f(a +) = f(a – 0). We shall very soon show you this. **(ii)** Usually, the first definition given above is used when the function is piecewise function and the second one is used otherwise.
Continuous functions may be thought of as those functions whose graphs can be drawn without lifting the pencil from a paper. Following figures show graphs of some functions which are discontinuous at x = a.





REMARK: First type of discontinuity is known as **"Jump Discontinuity"** and the second kind of discontinuity is known as **"Singularity"**.
For instance, function f(x) = 3x + 5 is continuous at x = 2 because: $\lim_{x\to 2} f(x) = f(2) = 11$

Some types of functions are continuous at every real number in their domain. Limits of such functions can always be determined by the substitution approach. Polynomial functions are continuous at every real number. Rational functions are continuous at every real number, except at numbers for which the denominator is zero.

Example 01: Is the function $f(x) = \dfrac{x^2-1}{x+1}$ **continuous at** x = 1?

Solution: Here $\lim_{x\to 1} f(x) = \lim_{x\to 1}\dfrac{x^2-1}{x+1} = \dfrac{1-1}{1+1} = 0$ and $f(1) = 0$. Since both limiting value and f(1) are equal and finite hence, f (x) is continuous at x = 1.

Example 02: Is the function $f(x) = \dfrac{x^2-1}{x+1}$ **continuous at x = -1?**

Solution: Since f(x) is undefined at x = -1, therefore, function f(x) is discontinuous at x = -1. However, $\lim_{x\to -1} f(x)$ does exist, since

$$\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x-1)(x+1)}{(x+1)} = \lim_{x\to -1}(x-1) = -2$$

-1 0 1

2

The graph of the function is shown here.
Since there is no point corresponding to x = -1, this discontinuity is called a "missing point discontinuity."

Example 03: Is the following function continuous at x = 0?

$$f(x) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

Solution: If the function has a point where it is discontinuous, the trouble spot will be at x = 0. Since the function is defined differently for x < 0 and $x \ge 0$, we must evaluate limits approaching zero from the left and from the right to test for continuity. Now, f(0) = 0 and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-}(0) = 0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$

Hence, given function is continuous at x = 0. This is shown in the above figure.

Example 04: Discuss the continuity of f defined by

$$f(x) \begin{cases} x-4 & -1 < x \le 2 \\ x^2-6 & 2 \quad x \quad 5 \end{cases}$$

Solution: Here, f(2) = 2 – 4 = -2. Also

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x-4) = -2,\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x^2-6\right) = -2\cdot$$

Since f(2 – 0) = f(2 + 0) = f(2) = -2, hence given function is continuous at x = 2.

Example 05: Examine the continuity the following function at x = 0

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution: Given that f(0) = 0. Moreover,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] = \lim_{x\to 0}(x)\cdot\lim_{x\to 0}\left[\sin\left(\frac{1}{x}\right)\right] = \lim_{x\to 0}(x)\cdot A = 0$$

Thus $\lim_{x\to 0} f(x) = f(0) = 0$. Hence given function is continuous at x = 0.

REMARK: **(i)** sin(1/x) is bounded function and its value lies between –1 and +1.

Example 06: Examine the continuity of following function at x = 0.

$$f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Solution: Here f(0) = 0 (Given). Also

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim_{x\to 0}\left[\frac{e^{1/x}\left(1-e^{-1/x}\right)}{e^{1/x}\left(1+e^{-1/x}\right)}\right] = \lim_{x\to 0}\left(\frac{1-1/e^{1/x}}{1+1/e^{1/x}}\right)$$

Applying the limit, we get

$$\lim_{x\to 0} f(x) = \frac{1-0}{1+0} = 1,\quad \left[\text{Note: } \lim_{x\to 0} e^{1/x} \to \infty \text{ and } \lim_{x\to 0}\frac{1}{e^{1/x}} \to 0\right].$$

Since $\lim_{x\to 0} f(x) \neq f(0)$, hence given function is discontinuous at x = 0.

Example 07: Examine the continuity of following function at x = 0.

$$f(x) = \begin{cases} \frac{\sin 5x}{\sin 3x} & x \neq 0 \\ 5/3 & x = 0 \end{cases}.$$

Solution: Here f(0) = 3/5 (Given). Also

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin 5x}{\sin 3x}\right) = \lim_{x\to 0}\left[\frac{\sin 5x}{5x}\times\frac{5x}{3x}\times\frac{3x}{\sin 3x}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\times\lim_{x\to 0}\frac{3x}{\sin 3x}\times\lim_{x\to 0}\frac{5x}{3x} = 1.1.\frac{5}{3} = \frac{5}{3}$$

Since $\lim_{x\to 0} f(x) = f(0)$, hence given function is continuous at x = 0.

Example 08: Examine the continuity of following function at x = 0.

$$f(x) = \begin{cases} (1+x)^{1/x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Solution: Here f(0) = 1 (Given). Also

$$\lim_{x\to0} f(x) = \lim_{x\to0}(1+x)^{1/x} = \lim_{x\to0}\left[1+\frac{1}{x}.x+\frac{1}{2!}\frac{1}{x}\left(\frac{1}{x}-1\right)x^2+\frac{1}{3!}\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)x^3+...\right]$$

$$= \lim_{x\to0}\left[1+1.1+\frac{1}{2!}\frac{(1-x)x}{x^2}.x^2+\frac{1}{3!}\frac{(1-x)(1-2x)}{x^3}x^3+...\right]$$

$$= \lim_{x\to0}\left[1+1+\frac{1}{2!}(1-x)+\frac{(1-x)(1-2x)}{3!}+...\right] = \left[1+1+\frac{1}{2!}+\frac{1}{3!}+...\right] = e$$

Since $\lim_{x\to0} f(x) \neq f(0)$, hence given function is not continuous at x = 0.

Example 09: Is function $f(x) = \lfloor x \rfloor$ continuous at x = 1.5?

Solution: (i) $f(1.5) = 1.5 = 1$ (ii) $\lim_{x\to1.5^-} f(x) = \lim_{x\to1.5^-} x = 1$, $\lim_{x\to1.5^+} f(x) = \lim_{x\to1.5^+} x = 1$

Since $\lim_{x\to1.5^-} f(x) = \lim_{x\to1.5^+} f(x) = 1$ $\therefore \lim_{x\to1.5} f(x) = 1$. Also, $\lim_{x\to1.5} f(x) = f(1.5) = 1$.

Hence, f (x) is continuous at x = 1.5.

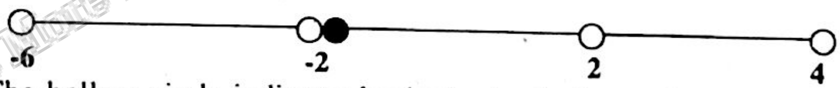
REMARK: The bracket function (the least integer function) is continuous at all real numbers except the points where x is an integer.

Example 10: Is the following function continuous?

$$f(x) = \begin{cases} x+4 & -6 < x < -2 \\ -x & -2 \le x < 2 \\ x-4 & 2 < x < 4 \end{cases}$$

Solution: You may observe that given function is piecewise function. Here, it is not given that where the continuities are to be determined. In such problems, points of discontinuities may occur at the intermediate points of the entire interval.

Here the entire interval (-6, 4) is divided in three parts (-6, -2), [-2, 2) and, (2, 4). This is depicted in the following figure. Hence, points of discontinuities may occur at x = -2 and x = 2.



REMARK: The hollow circle indicates beginning/end of open interval and thick circle indicates beginning/end closed interval.

Let us now examine where the function is continuous and where it is discontinuous.

Continuity at x = -2

Consider, $f(-2-0) = \lim_{x\to-2^-} f(x) = \lim_{x\to-2}(x+4) = 2$

And $f(-2+0) = \lim_{x\to-2^+} f(x) = \lim_{x\to-2}(-x) = 2$. Also $f(-2) = -(-2) = 2$

Since f(-2 – 0) = f(-2 + 0) = f(-2) = 2, hence give function is continuous at x = -2.

Continuity at x = 2

Consider, $f(2-0) = \lim_{x\to2^-} f(x) = \lim_{x\to2}(-x) = -2$

And $f(2+0) = \lim_{x\to2^+} f(x) = \lim_{x\to2}(x-4) = -2$.

You may observe that left and right limits are equal but f(2) is not defined. Thus given function is not continuous at x = 2.

Example 11: If the following function is continuous find the values of unknowns `a` and `b`.

$$f(x)=\begin{cases} x^3 & x<-1 \\ ax+b & -1\le x<1 \\ x^2+2 & x\ge 1 \end{cases}$$

Solution: The intermediate points are -1 and +1. Now let us consider:

$f(-1-0)=\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1}\left(x^3\right)=-1$.

And $f(-1+0)=\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1}(ax+b)=-a+b$

Now consider, $f(1-0)=\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1}(ax+b)=a+b$

And $f(1+0)=\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1}\left(x^2+2\right)=3$

It is given that function is continuous, hence

$f(-1-0)=f(-1+0)$ ➔ $-1=-a+b$.

Also, $f(1-0)=f(1+0)$ ➔ $a+b=3$.

Solving these equations simultaneously, we get: a = 2 and b = 1.

Thus for a = 2 and b = 1given function is continuous.

$\in-\delta$ Definition of Continuity

A function f(x) is said to be continuous at x = a if for given $\in>0$ there exists a real number $\delta>0$ such that $|f(x)-f(a)|<\in$ whenever $|x-a|<\delta$.

We shall use the above definition of continuity to see how it is applied to see whether or not a given function is continuous or not.

Example 12: Using $\in-\delta$ definition, show that f(x) = 2 x – 3 is continuous at x = 4.

Solution: Here f(x) = 2x – 3, thus f(a) = f(4) = 8 – 3 = 5. Now

$|f(x)-f(a)|=|2x-3-5|=|2x-8|=2|x-4|$

Now let $|x-a|=|x-4|<\delta$.

This implies that

$|f(x)-f(a)|=|f(x)-f(4)|=2|x-4|=2\delta<\in$, where $\in=2\delta$.

This proves that given function is continuous at x = 4.

Example 13: Use $\in-\delta$ definition, to show that f(x) = x^2-3 is continuous at x = 2.

Solution: Here f(x) = x^2-3, thus f(a) = f(2) = 4 – 3 = 1. Now

$|f(x)-f(a)|=|x^2-3-1|=|x^2-4|=|(x-2)(x+2)|=|x-2|\,|x+2|$

Now let $|x-a|=|x-2|<\delta$

This implies that: $|f(x)-f(a)|=|f(x)-f(4)|=|x-2||x+2|=\delta|x+2|<\in$, where $\in=\delta|x+2|$. This proves that given function is continuous at x = 2.

Applications of Continuity

In this section we shall present the applications of limit and continuity that will help the readers to understand the areas where the concepts of continuity are used.

Example 14: An ant trap is used to eliminate an ant colony. The number of ants living t hours after the trap is put out is given by

$$N(t)=\begin{cases} 2040-5t & 0\le t<24 \\ 2880-40t & 24\le t\le 72 \end{cases}$$

(a) How many ants were originally present in the colony?

(b) How many ants are left after 72 hours?

(c) Is the elimination of ants continuous at t = 24 hours? Explain.

Solution: (a) The number of ants originally present in the colony was

$$N(0) = 2040 - 5(0) = 2040$$

(b) The number of ants left after 72 hours is

$$N(72) = 2880 - 2880 = 0$$

(c) The function will be continuous at t = 24 if

$$\lim_{t\to 24^-} N(t) = \lim_{t\to 24^+} N(t) = N(24).$$

Now,
$$\lim_{t\to 24^-} N(t) = \lim_{t\to 24^-} (2040-5t) = 2040-120 = 1920$$

$$\lim_{t\to 24^+} N(t) = \lim_{t\to 24^+} (2880-40t) = 2880-960 = 1920$$

Also,
$$N(24) = 2880-40(24) = 2880-960 = 1920$$

Now N(24 – 0) = N(24 + 0) = N(24) = 1920. This implies that elimination of ants is continuous at t = 24 hours.

Example 15: The flow of current is given as under:

$$I(t) = \begin{cases} t^2+1 & 0 \le t < 1 \\ 2t & 1 < t < 2 \\ t^2 & 2 \le t < 4 \end{cases}$$

where t is the time in seconds. Is the current flow continuous at t = 1, t = 2? What will be the value of current at t = 0 and t = 3.5 and t = 4 seconds?

Solution: Consider: $I(1-0) = \lim_{t\to 1-0} I(t) = \lim_{t\to 1}(t^2+1) = 2$

Also
$$I(1+0) = \lim_{t\to 1+0} I(t) = \lim_{t\to 1}(2t) = 2$$

Since left and right limits at t = 1 are equal but I(1) is not defined. Hence flow of current cannot be continuous at t = 1 sec.

Now consider:
$$I(2-0) = \lim_{t\to 2-0} I(t) = \lim_{t\to 2}(2t) = 4$$

Also
$$I(2+0) = \lim_{t\to 2+0} I(t) = \lim_{t\to 2}(t^2) = 4$$

And
$$I(2) = 2^2 = 4$$

Since I(2 – 0) = I(2 + 0) = I(2) = 4, hence flow of current is continuous at t = 2 seconds. Now at t = 0, $I(0) = 0^2 + 1 = 1$ Ampere. At t = 3.5, $I = (3.5)^2 = 12.25$ A. Finally, since t = 4 does not come in the domain of current function, hence we can not say any thing about the flow of current when time t = 4 sec. In other words, at t = 4 seconds the current value is not defined.

Example 16: A colony of bacteria is introduced to a growth inhibiting environment and grows at time t according to the formula:

$$B(t) = \begin{cases} t+8 & -6 \le t < -2 \\ t^2+2 & -2 \le t < 2 \\ 5t & 2 \le t < 6 \end{cases}$$

where B is in thousands. The negative sign shows the time before colony of bacteria was seen. [For instance B(-6) means number of bacteria in the colony six hours ago were -6 + 8 = 2 thousands]. Is the growth of bacteria continuous at t = -2 and t = 2?

Solution: The given function will be continuous at t = 2, if

$$\lim_{t\to -2-0} B(t) = \lim_{t\to -2+0} B(t) = B(-2)$$

Now $\lim_{t\to -2-0} B(t) = \lim_{t\to -2-0}(t+4) = 6$ and $\lim_{t\to -2+0} B(t) = \lim_{t\to -2+0}(t^2+2) = 6$. Also $B(-2) = 6$

Since, B(-2 – 0) = B(-2 + 0) = B(-2) = 6. Hence, Bacteria growth is continuous at t = -2 (Two hours before now)

Also, $\lim_{t\to 2-0} B(t) = \lim_{t\to 2-0}(t+4) = 6$ and $\lim_{t\to 2+0} B(t) = \lim_{t\to 2+0}(5t) = 10$

Since, left and right limits are not equal at t = 2 hence, Bacteria growth is not continuous at t = 2 (Two hours after now).

WORKSHEET 02

1. Evaluate the following limits:

(a) $\lim_{x\to\infty} \dfrac{x^4 - 3x}{x^5 + x^2 - 9}$ (b) $\lim_{x\to 1+0} \dfrac{x-1}{\sqrt{x^2-1}}$ (c) $\lim_{x\to 0} \dfrac{|-1+x|-1}{|x|}$ (d) $\lim_{x\to 3-0} \dfrac{3-x}{|x-3|}$

2. Suppose the cost C of removing p percent of the particulate pollution from the smokestacks of an industry plant is given by: $C(p) = \dfrac{7300\ p}{100 - p}$. Find $\lim_{p\to\infty} C(p)$.

3. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a commercial customer is given by the following function:

$$C(x) = \begin{cases} 7.52 + 0.1079x, & 0 \le x \le 5 \\ 19.22 + 0.1079x, & 5 < x \le 750 \\ 20.795 + 0.1058x, & 750 < x \le 1500 \\ 131.345 + 0.0321x, & x > 1500 \end{cases}$$

Find (a) Find $\lim_{x\to 1500} C(x)$ and (b) $\lim_{x\to 5} C(x)$.

4. The monthly charge for water in a small town is given by:

$$f(x) = \begin{cases} 18, & 0 \le x \le 20 \\ 18 + 0.1(x-20), & x > 20 \end{cases}$$

Find $\lim_{x\to 20} f(x)$.

5. The profit from the sale of x units is: $P(x) = \dfrac{1400x - 250}{x}$ dollars for $x \ge 1$

What is the limit of profit as the quantity sold increases without bound?

6. The amount of a drug that remains in a person's bloodstream t hours after being injected is given by: $f(t) = 0.15t/(1 + t^2)$. Find $\lim_{t\to\infty} f(x)$ if it exists. Explain the result you obtain.

7. Function f(x) is defined as:

$$f(x) = \begin{cases} \dfrac{(x^3 - 27)}{(x-3)} & x \ne 3 \\ 9/2 & x = 3 \end{cases}$$

Is f (x) continuous at x = 3?

8. Is the following function continuous?

$$f(x)=\begin{cases} x+2 & 0<x<1 \\ x & 1\le x<2 \\ x+5 & 2\le x<3 \end{cases}$$

9. Examine the continuity of following function at x = 0.

(a) $f(x)=\begin{cases} \dfrac{e^{1/x}}{e^{1/x}+1} & x\neq 0 \\ 0 & x=0 \end{cases}$ (b) $f(x)=\begin{cases} \dfrac{\sin 7x}{\sin 6x} & x\neq 0 \\ 6/7 & x=0 \end{cases}$

(c) $f(x)=\begin{cases} (1+4x)^{1/x} & x\neq 0 \\ e^4 & x=0 \end{cases}$

10. Use $\in-\delta$ definition, show that $f(x) = x^3 - 3$ is continuous at x = 4.

11. In 1986, Tax Reform Act created the following tax schedule for a single filers in 1988.

Tax Rate	Tax Income
15%	0 - $17850
28%	Above $17850

This means that the tax T is a function of income x as follows:

$$T(x)=\begin{cases} 0.15x, & 0\le x\le 17850 \\ 2677.50+0.28(x-17850), & x>17850 \end{cases}$$

Is T(x) continuous?

12. Residential customers in a small town have their monthly charge f(x) for x hundred gallons of water given by:

$$f(x)=\begin{cases} 18, & 0\le x\le 20 \\ 18+0.1(x-20), & x>20 \end{cases}$$

Is f(x) continuous?

13. Suppose that cost C of removing p percent of particulate pollution from the exhaust gases at an industrial site is given by: C(p) = 8100 p/(100 – p).Describe any discontinuity for C (p).

14. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a commercial customer is given by the following function:

$$C(x)=\begin{cases} 7.52+0.1079x, & 0\le x\le 5 \\ 19.22+0.1079x, & 5<x\le 750 \\ 20.795+0.1058x, & 750<x\le 1500 \\ 131.345+0.0321x, & x>1500 \end{cases}$$

a. What will be the charges for using the 5 and 750 KWH of electricity?
b. Is the function continuous at x = 5 and x = 15--?

15. Suppose the size of a population of bacteria in time t is given by:

$$P(t)=\begin{cases} t, & \text{when } 0\le t\le 1/2 \\ 1, & \text{when } t=1/2 \\ 1-t, & \text{when } 1/2<t<1 \end{cases}$$

a. How many bacteria were originally present?
b. Is the function continuous at t = 1/2 hours?

16. The amount of a drug that remains in a person's bloodstream t hours after being injected is given by:

$$f(t) = \begin{cases} t+2, & \text{when } 0 \le t \le 1 \\ t, & \text{when } 1 \le t < 2 \\ t+5, & \text{when } 2 \le t < 3 \end{cases}$$

a. What was the amount of drug at the beginning?
b. Is the function continuous at t = 2 and t = 3 hours?

17. Suppose the cost of obtaining water that contains $p\%$ impurities is given by:

$$C(p) = \begin{cases} \dfrac{120,000}{p} - 1200, & \text{if } 0 < p \le 10 \\ 10800, & \text{if } 10 < p \le 12 \end{cases}$$

a. What is the cost of obtaining water that contains 12% impurities?
b. What is the cost of obtaining water that contains 5% impurities?
c. Is the function continuous at p = 10%?

18. The velocity V(t) m/sec of a particle is given as:

$$V(t) = \begin{cases} t, & \text{when } t \le 0 \\ t^2, & \text{when } 0 < t \le 4 \\ 2t+4, & \text{when } t > 4 \end{cases}$$

a. What was the velocity at the beginning?
b. Is the velocity of the particle continuous at t = 0 and t = 4?

19. Suppose that the cost to remove x percent of the pollutants in a lake is given by:

$$C(x) = \begin{cases} \dfrac{900,000}{100-x}, & \text{if } 0 < x \le 20 \\ 11250, & \text{if } x > 20 \end{cases}$$

a. What is the cost to remove 10% of the pollutants from the lake?
b. Is the function continuous at x = 20%?

CHAPTER THREE

DERIVATIVES

3.1 WHAT IS CALCULUS

Calculus is the mathematics of motion and change. Where there is motion or growth, where forces are at work producing acceleration, calculus is the right mathematics to apply. This was true in the beginnings of the subject and it is true today.

Calculus was first invented to meet the mathematical needs of the scientists of sixteenth and seventeenth centuries, needs that were mainly mechanical in nature. There are two main branches of calculus called **Differential Calculus** and **Integral Calculus**.

Differential calculus deals with the problem of calculating rates of change. It enabled people to define slopes of curves, to calculate velocities and accelerations of moving bodies, to find firing angles that would give cannons their greatest range, and to predict the times when planets would be closest together or farthest apart. Integral calculus deals with the problems of determining a function from information about its rate of change. It enables to calculate the future location of a body from its present position and knowledge of the forces acting on it, to find the areas of irregular regions in the plane, to measure the lengths of curves, and to find the volumes and masses of arbitrary solids.

Today, applications of calculus and its extensions in mathematical analysis are far reaching indeed, and the physicists, mathematicians and astronomers who first invented the subject would surely be amazed and delighted, as we hope you will be, to see what a large number of problems it solves and what a range of fields now use it in the mathematical models that bring understanding about the universe and the world around us.

Calculus is widely employed in physical, biological and social sciences. It is used, for example, in the physical sciences to study the speed of a falling body, the rates of change in a chemical reaction, or the rate of decay of a radioactive material. In biological sciences, a problem such as rate of growth of a colony of bacteria as a function of time is easily solved using calculus. In social sciences, calculus is widely used in the study of Statistics and Probability and optimization problems.

Calculus can be applied to many problems involving the notion of extreme values, such as the fastest, the most, the slowest or the least. These maximum or minimum amounts may be described as values for which a certain rate of change (increase or decrease) is zero. By using calculus it is possible to determine how high a projectile will go by finding the point at which its change of altitude with respect to time, that is, its velocity, is equal to zero.

The invention of calculus had a great impact on technology as well as on the development of mathematics. Years later, applications of calculus were found in a variety of non–engineering areas, including business and economics, biology, medicine, sociology and psychology. Calculus can be used to:

- Determine the average speed at which blood flows through an artery.
- Select the most economical dimensions for packaging.
- Calculate how high a projectile will travel.
- Find the production level that will maximize a company's profit.

Contributions of Mathematicians to Calculus

The Englishman **Isaac Newton** (1642–1727) and the German **Gottfried Wilhelm Leibniz** (1646–1716) are the mathematicians credited with inventing calculus. They worked independently of each other. Newton invented calculus in 1665 but took more than 20 years to publish his results hence Leibniz's development of calculus was published first. Furthermore, Leibniz's notation was considered superior to Newton's notation, and it is still used today.

Sir Isaac Newton n

Gottfried Wilhelm Leibniz

How to Learn Calculus

Learning calculus is not the same as learning arithmetic, algebra and geometry. In these subjects, you learn primarily how to calculate with numbers, how to simplify algebraic expressions and calculate with variables and how to reason about the points and lines and figures in the plane. Calculus involves those techniques and skills but develops others as well, with greater precision and a deeper level. Calculus introduces so many new concepts and computational operations, in fact, that you will no longer be able to learn everything you need in class. You will have to learn a fair amount on your own by working with other students.

- **Read the text:** You will be able to learn all meanings and connections you need just by attempting the exercises. You will need to read relevant passages in the book and work through examples step-by-step. Speed-reading will not work here. You are reading and searching for detail in a step-by-step logical fashion. This kind of reading, required by any deep and technical content, takes attention, patience and practice.
- **Do the given work in time, keeping the following principles in mind:**
  - Sketch diagrams whenever possible.
  - Write your solutions in a connected step-by-step logical fashion, as if you were explaining to someone else.
  - Think about why each exercise is there? Why was it assigned? How it is related to the other assigned exercises?
- **Use your calculator and computer whenever possible:** Graphs provide insight and visual representations of important concepts and relationships.
- **Try to write your own:** Write short descriptions of the key points each time you complete a section of the text. If you succeed, you probably understand the material. If you do not, you will know where there is a gap in your understanding.

3.2 DERIVATIVES

Before proceeding to a formal definition of derivative, consider some background information on the nature of derivative. Simply stated, the derivative is a rate of change. Three examples are given to offer some preliminary insight into the notion of rate of change:

- Consider an outbreak of flu. A function specifies the number of people sick due to flu at any particular time. The derivative of the function indicates the rate at which illness due to flu is spreading at any particular time.

- Suppose that a function gives the average cost per unit of producing x units. The derivative of the function gives information about when the average cost per unit is increasing and when it is decreasing.
- A function may describe the motion of a rocket, giving the distance traveled for any time t. The derivative of this function is the rate of change of distance with respect to time–the velocity. Using the derivative, we can determine the velocity at any instant desired.

Derivatives are also used to find the slope of a tangent line to curve $y = f(x)$ at any point. For example, what would be the slope of the tangent line to the curve $y = x^2 + 1$ at $x = 2$? As mentioned above that major techniques of **CALCULUS** are Differentiation and Integration or anti-derivative. The part of calculus that is associated with differentiation is called **Differential Calculus** and the part of calculus that involves integration is known as **Integral Calculus**. The main objectives of differential calculus are to establish the measure of the changes in a particular function with mathematical accuracy. "Differentiation is a process of finding the rate at which one variable quantity changes with respect to another.

Increment of a Function

Literally the word increment means an increase; but in MATHEMATICS increment means small change in the value of a variable. Increment of a variable may be positive or negative. The increment in the variable x is denoted by Δx or δx or h. Thus,

Δx = increment in x $\qquad$ Δy = increment in y

Procedure to Find the Derivative of a Function

Let y be a function of an independent variable x, that is, let

$$y = f(x) \qquad (1)$$

Let there be an increment /change Δx in x and the corresponding change in y be Δy. Thus equation (1) becomes,

$$y + \Delta y = f(x + \Delta x) \qquad (2)$$

Subtract (1) from (2), we get:

$$\Delta y = f(x + \Delta x) - f(x) \qquad (3)$$

Dividing (3) by Δx to get the quotient:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (4)$$

This ratio is called the incremental ratio OR the average rate of change of f(x) over the interval, $[x, x + \Delta x]$. Now taking limit Δx to zero, we obtain:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (5)$$

This limit, when it exists, is denoted by $f'(x)$ and is called the **"Derivative"** or **"Differential Coefficient"** of $y = f(x)$ with respect to x. This is also called the **"Instantaneous Rate of Change"** or **"Slope of the Curve"** $y = f(x)$ at x. The method described above for finding the derivative of the function f(x) is called **"First principle"** or **"ab-initio"** method. It is also known as derivative by definition". Thus, (5) can be written as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Other symbols used for the derivative of $y = f(x)$ are: y`, D y, dy / dx.

REMARK: The notation dy/dx is the originally by Leibniz.

Example 01: Find the derivatives of the following functions by definition.

(i) $y = x$ $\qquad$ (ii) $y = 1/x$ $\qquad$ (iii) $y = x^n$

Solution: (i) Given y = x. Let Δx be the change in x and Δy the corresponding change in y. Then,

$$y+\Delta y = x+\Delta x \Rightarrow \Delta y = x+\Delta x - y = x+\Delta x - x \Rightarrow \Delta y = \Delta x$$

Dividing both sides by Δx, we get: $\frac{\Delta y}{\Delta x}=1$

Taking the limit $\Delta x \to 0$, to obtain: $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=1$.

Since the limit exists (finite), hence, $dy/dx = 1$

(ii) We have $y=\frac{1}{x} \Rightarrow y+\Delta y=\frac{1}{(x+\Delta x)} \Rightarrow \Delta y=\frac{1}{(x+\Delta x)}-y$

$$\Rightarrow \quad \Delta y=\frac{1}{(x+\Delta x)}-\frac{1}{x}=\frac{x-(x+\Delta x)}{x(x+\Delta x)}=\frac{-\Delta x}{x(x+\Delta x)}$$

Dividing both sides by Δx and taking the limit $\Delta x\to 0$, we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left[\frac{-1}{x(x+\Delta x)}\right]=-\frac{1}{x^2} \qquad \Rightarrow \frac{dy}{dx}=-\frac{1}{x^2}$$

(iii) Given that $y=x^n$

$\Rightarrow y+\Delta y=(x+\Delta x)^n \qquad \Rightarrow \Delta y=(x+\Delta x)^n - y$

$$\Rightarrow \Delta y = x^n + nx^{n-1}\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots+(\Delta x)^n - x^n$$

$$= nx^{n-1}\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots+(\Delta x)^n$$

Dividing both sides by Δx and taking the limit $\Delta x\to 0$, to obtain

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x\left[nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}\Delta x+\ldots+(\Delta x)^{n-1}\right]}{\Delta x}=nx^{n-1}+0+..+0$$

Since the limit exists (finite) hence, $dy/dx = nx^{n-1}$

Thus if: $y=x^n \Rightarrow dy/dx = nx^{n-1}$

One Sided Derivative

Recall that the function y = f(x) is said to be "Derivable" or "Differentiable" at x if the limit $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists.

Equivalently, we say that a function is derivable if the one – sided limits

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} \text{ and } \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$

exist and are equal. (Note: $h=\Delta x$). If these limits exist, they are respectively denoted by $Lf'(x)$ and $Rf'(x)$. If $Lf'(x)$ and $Rf'(x)$ exist but are not equal, that is; $Lf'(x)\neq Rf'(x)$, we say that the function f(x) possesses left hand and right hand derivatives but is not derivable or differentiable.

Derivative at a Point

It may be noted that if the derivative of f(x) is required at some point say at x = a, where $a\in$ Dom(f) the derivative is denoted by

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}.$$

Thus by definition, $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

Writing a + h = x in the above equation, we have

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}, \text{ since } x \to a \text{ as } h \to 0.$$

This is another way to represent the derivative of f(x) at x = a.

For instance, derivative of $y = x^2$ at x = 3 may be found by using ab-initio method as:

$$f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0}\frac{9+6h+h^2-9}{h}$$

$$= \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6$$

REMARK: If a function f(x) is differentiable, it must be continuous. But the converse is not true, that is, if a function is continuous it may not be differentiable.

We provide an example that will make this idea clear to readers.

Example 02: Let f(x) = | x | from R to R. Discuss the continuity and differentiability of f at x = 0.

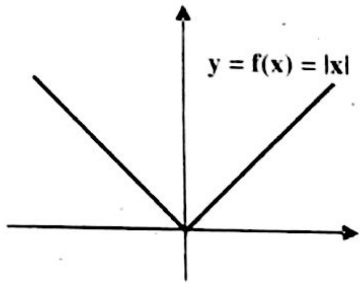
Solution: Continuity at x = 0:

Here, f(0) = |0| = 0. Also,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-}(-x) = 0 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+}(x) = 0$$

Since, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) \Rightarrow \lim\limits_{x\to 0} f(x) = 0$

Thus, $\lim\limits_{x\to 0} f(x) = f(0) = 0$



Hence, f(x) = | x | is continuous at x = 0.

Differentiability at x = 0:

By definition,

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \Rightarrow Lf'(a) = Rf'(a)$$

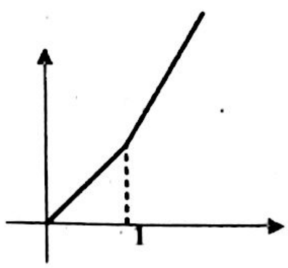
Since x = 0, we have

$$Lf'(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|x|-0}{x} = \lim_{x\to 0^-}\frac{(-x)}{x} = \lim_{x\to 0^-}(-1) = -1,$$

$$Rf'(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{|x|-0}{x} = \lim_{x\to 0^+}\frac{(x)}{x} = \lim_{x\to 0^+}(1) = +1.$$

Since, $Lf'(0) \neq Rf'(0)$, therefore, f(x) = |x| is not differentiable at x = 0 although it is continuous there at.

Example 03: Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x-1 & 1 < x < 2 \end{cases}$



Discuss the continuity and differentiability of f at x = 1.

Solution: Continuity at x = 1:

Here, f(1) = 1. Also,

$f(1^{-1}) = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x) = 1$, and $f(1^+)\ \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2x-1) = 1$

Since, $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) \Rightarrow \lim_{x\to1} f(x) = 1$. Hence, f(x) is continuous at x = 1.

Differentiability at x = 1:

By definition, $f'(a) = \lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$. Since x = 1, we have

$$Lf'(1) = \lim_{x\to1^-}\frac{f(x)-f(1^-)}{x-1} = \lim_{x\to1^-}\frac{x-1}{x-1} = \lim_{x\to1^-}(1) = 1$$

$$Rf'(1) = \lim_{x\to1^+}\frac{f(x)-f(1^+)}{x-1} = \lim_{x\to1^+}\frac{(2x-1)-(1)}{x-1} = \lim_{x\to1^+}\frac{(2x-2)}{(x-1)} = \lim_{x\to1^+}\frac{2(x-1)}{(x-1)} = 2$$

Since, $Lf'(1) \neq Rf'(1)$, therefore, f(x) is not differentiable at x = 1.

Example 04: Find the values of a and b so that the function f is continuous and differentiable at x = 1 where

$$f(x) = \begin{cases} x^3 & x < 1 \\ ax + b & x \geq 1 \end{cases}$$

Solution: Continuity at x = 1:

Here, f(1) = a(1) + b = a + b. Also

$f(1-) = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^3) = 1$, $f(1+) = \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(ax+b) = a+b$

For a function to be continuous, we must have

f(1 – 0) = f(1 + 0) ➔ a + b = 1 (1)

Differentiability at x = 1:

$$Lf'(1) = \lim_{x\to1^-}\frac{f(x)-f(1^{-1})}{x-1} = \lim_{x\to1^{-1}}\frac{x^3-1^3}{x-1} = \lim_{x\to1^{-1}}\frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to1^{-1}}(x^2+x+1) = 3,$$

$$Rf'(1) = \lim_{x\to1^+}\frac{f(x)-f(1^{+1})}{x-1} = \lim_{x\to1^{+1}}\frac{(ax+b)-(a+b)}{x-1} = \lim_{x\to1^{+1}}\frac{(ax-a)}{x-1} = \lim_{x\to1^{+1}}\frac{a(x-1)}{(x-1)} = a$$

For a function to be derivable, we must have: a = 3

Substituting it into (1), we get: $3 + b = 1 \Rightarrow b = -2$

Hence, for a = 3 and b = -2, the given function is continuous as well as derivable.

Geometrical Meaning of Derivative

Let f(x) be a differentiable function given by the equation y = f (x). Let the graph of this function be shown as under. Let P(x, y) and Q(x + Δx, y + Δy) be two distinct points on this curve. If θ is the angle that the secant line PQ makes with the x–axis, then

$$\tan\theta = \frac{QS}{PS} = \frac{(y+\Delta y)-y}{(y+\Delta x)-x} = \frac{\Delta y}{\Delta x} \qquad (1)$$

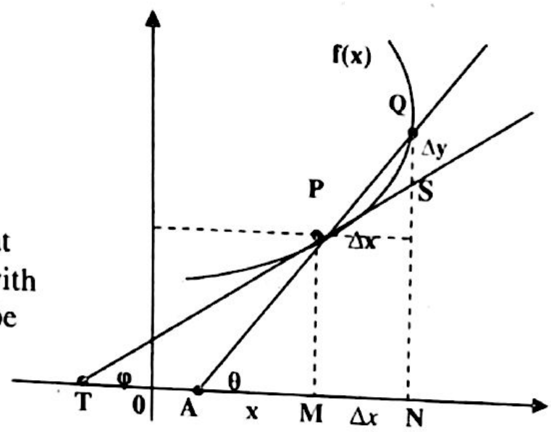
This can also be expressed as

$$\tan\theta = m_{sec} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ [m_{sec} means the slope of the secant line.]

Thus, $m_{sec} = \Delta y/\Delta x$, slope of the secant line APQ. As Δx gets smaller, the secant line comes closer and to closer to tangent line at P. As Δx approaches zero, the secant line approaches the tangent line. This means that the slope of the tangent line is the limit of the slope of the secant line as Δx approaches zero. Hence taking limit Δx→0, equation (1) becomes:

$$\lim_{Q\to P} m_{sec} = \lim_{Q\to P}\frac{\Delta y}{\Delta x} = m_{tan} = \frac{dy}{dx}$$

Thus, derivative of the function f at the point P represents the slope of the tangent line to the curve $y = f(x)$ at P.



In the rectangular coordinates system, the rate at which y–coordinate of a straight line changes with respect to the x–coordinate is known as the slope of the line.

Thus the slope is the rate of change of y with respect to x.

$$\text{Slope}(m) = \frac{\text{change in y- coordinate}}{\text{change in x- coordinate}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Here (x_1, y_1) and (x_2, y_2) are the coordinates of the points P and Q respectively. The slope of a line is a number that specifies the change in y compared with the change in x in going from point to point on the line. For any particular line, the slope is constant for the entire line. For example, $y = f(x) = 3x + 2$ has the slope +3 because y increases by three units for every unit increase in x. Let us see this:

x:	-2	-1	0	1	2
y:	-4	-1	2	5	8

Now take any two values of x and the corresponding values of y, say x = -1 and 1. The corresponding values of y are -1 and 5. By definition:

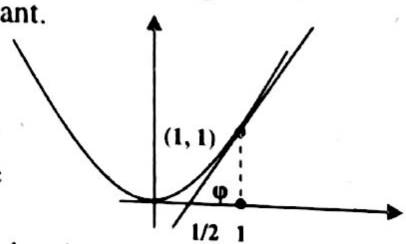
$$\text{Slope}(m) = \frac{\text{change in y- coordinate}}{\text{change in x- coordinate}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{5-(-1)}{1-(-1)} = \frac{6}{2} = 3$$

Now let us take other two values of x say x = 1 and 2. The corresponding values of y are 5 and 8. Using the formula, we get

$$\text{Slope}(m) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-5}{2-1} = 3$$

This is same as before. Thus no matter what two values of x and the corresponding values of y are considered, the slope of a given line always remains constant.

But what happens when the function under consideration is not linear? In order to consider the rate of change in such instances, we might choose to extend the notion of slope to curves other than straight lines. We have already said that slope of curve is the slope of tangent line at a particular point P.



For example, consider $y = x^2$, then $y' = 2x$. At x = 1, $y' = 2$. Now the slope of the tangent to the curve $y = x^2$ at (1, 1) is

$$\tan\varphi = \frac{\text{perp}}{\text{base}} = \frac{1}{1/2} = 2$$

We observe that both results are same.

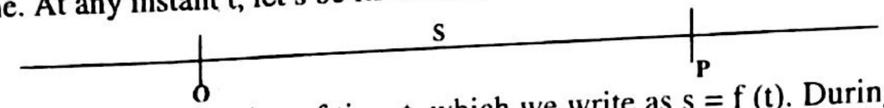
Example 05: Let $f(x) = x^3 - 5x^2 + 7$. Find $f'(1)$ and $f'(2)$. For what value of x the slope of this curve is zero?

Solution: Given that $f'(x) = x^3 - 5x^2 + 7$. Differentiating w.r.t x, we get $f'(x) = 3x^2 - 10x$. Thus $f'(1) = 3.1^2 - 10(1) = -7$. Similarly, $f'(2) = 3.2^2 - 10(2) = -8$. Now if the slope of the curve is zero, we have:

$3x^2 - 10x = 0 \rightarrow x(3x - 10) = 0 \rightarrow x = 0$ and $x = 10/3$.

Physical Meaning of Derivative

At the beginning of the chapter, we mentioned some problems of rate of change of different quantities with respect to time. This rate of change with respect to time is known as the derivative. There are enormous applications of derivatives where the rate of change is used. Let us elaborate this in some more detail. Consider a particle moving along a straight line. At any instant t, let s be its distance from a fixed point 'O' on the line.



Then the distance s is a function of time t, which we write as $s = f(t)$. During the interval $[t_1, t_2]$, it travels a distance $f(t_2) - f(t_1)$. The ratio

$$\left[f(t_2) - f(t_1)\right] / (t_2 - t_1)$$

gives the average speed during the interval $[t_1, t_2]$. Now the particle may be moving faster at some point and slower at others. Given any point t_0, the velocity at t_0 can be approximated by the average velocity in a small interval containing t_0. Then,

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

is called the **"instantaneous velocity"** at t_0, or the rate of change of s with respect to t at the instant t_0. Thus, velocity v at the instant t as t approaches t_0 is:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(\Delta t + t) - f(t)}{\Delta t} = \frac{ds}{dt} = f'(t)$$

The rate of change of velocity is known as acceleration and is defined as:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{f'(\Delta t + t) - f'(t)}{\Delta t} = \frac{dv}{dt} = f''(t)$$

Example 06: A body moves horizontally and its position (in feet) at time t (seconds) is $s = t^3 - 6t^2 + 9t$. Find the body's acceleration at time when its velocity is zero. Also find the acceleration when t = 3 sec.

Solution: We have $s(t) = t^3 - 6t^2 + 9t$. Differentiating with respect to t, we get

$$v = \frac{d}{dt}(s(t)) = \frac{d}{dt}(t^3 - 6t^2 + 9t) = 3t^2 - 12t + 9.$$

Again differentiating: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(3t^2 - 12t + 9) = 6t - 12$ (1)

Velocity of the body will be zero if

$$3t^2 - 12t + 9 = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t^2 - t - 3t + 3 = 0 \Rightarrow (t-1)(t-3) = 0 \Rightarrow t = 1, 3$$

Thus, acceleration of the body at t = 1: $a = 6(1) - 12 = -6$ feet/sec^2.

The acceleration of the body at t = 3 is: $a = 6(3) - 12 = 6$ ft/sec^2.

3.3 RULES OF DERIVATIVES

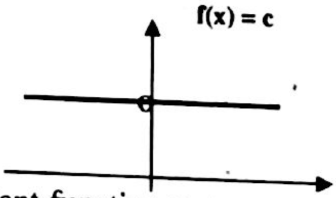
Finding the derivatives of functions by definition, that is,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is a lengthy and difficult procedure specially when given function involves combination of algebraic and transcendental functions. Rules of derivatives have been developed to make this process of finding the derivatives in an easy and simple way. Proofs of these may be found in any elementary calculus book.

1. **Constant Rule:** The derivative of a constant function is zero, that is; $\frac{d}{dx}(c)=0,\ c\in R$.

For example, $\frac{d}{dx}(5)=0, \frac{d}{dx}(-10)=0$ and so on.



REMARK: Readers are familiar with the fact that a constant function y = c represents a straight line parallel to x-axis and the slope of such straight line is zero.

2. **Power Rule:** If $f(x)=x^n$, where $n\in R$, then $f'(x)=nx^{n-1}$. For example,

$$\frac{d}{dx}\sqrt[3]{x}=\frac{d}{dx}\left(x^{1/3}\right)=\frac{1}{3}x^{1/3-1}=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}},\ \text{and}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}\left(x^{-1}\right)=-1\left(x^{-1-1}\right)=-1\left(x^{-2}\right)=\frac{-1}{x^2}.$$

3. **Coefficient Rule:** If $f(x)=c\cdot u(x)$, where $c\in R$, then $f'(x)=c\cdot u'(x)$.

For example: $\frac{d}{dx}\left(5x^3\right)=5\cdot\frac{d}{dx}\left(x^3\right)=5\left(3x^2\right)=15x^2$.

4. **Sum Rule:** If $f(x)=u(x)\pm v(x)$, where u and v are functions of x, then

$$f'(x)=u'(x)\pm v'(x).$$

For example: $\frac{d}{dx}\left(9x^2-3x\right)=9\frac{d}{dx}\left(x^2\right)-3\frac{d}{dx}(x)=9(2x)-3.1=18x-3$.

5. **The Product Rule:** If f(x) = u(x) v(x) where u and v are functions of x, then:

$f'(x)=u(x)v'(x)+v(x)u'(x)$. For example:

$$\frac{d}{dx}\left(x^2\cdot x^{1/3}\right)=\left(x^2\right)\cdot\frac{d}{dx}\left(x^{1/3}\right)+\left(x^{1/3}\right)\cdot\frac{d}{dx}\left(x^2\right)=\left(x^2\right)\left(\frac{1}{3x^{2/3}}\right)+\left(x^{1/3}\right)(2x)=\frac{1}{3}x^{4/3}+2x^{4/3}.$$

6. **The Quotient Rule:** If f(x) = u(x)/v(x) where u and v are differentiable functions of x with $v(x)\neq 0$, then $f'(x)=\frac{v(x)\cdot u'(x)-u(x)\cdot v'(x)}{\left[v(x)\right]^2}$.

For example: $\frac{d}{dx}\left(\frac{x^2-4x}{x+5}\right)=\frac{(x+5)(2x-4)-\left(x^2-4x\right)(1)}{(x+5)^2}$

$$=\frac{2x^2-4x+10x-20-x^2+4x}{(x+5)^2}=\frac{x^2+10x-20}{(x+5)^2}$$

7. **The Chain Rule:** The chain rule is defined in the following way:

If y = f(u) and u = g(x) then: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

For example if $y=\sqrt{\frac{1+x}{1-x}}$ then by using chain rule we may find its derivative as under by using chain rule.

Let $u=\frac{1+x}{1-x}$, then $y=\sqrt{u}=(u)^{1/2}$. Then, by chain rule: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

$$\frac{dy}{du}=\frac{d}{du}\left(u^{1/2}\right)=\frac{1}{2}\left(u^{-1/2}\right)=\frac{1}{2\sqrt{u}}=\frac{1}{2}\sqrt{\frac{1-x}{1+x}}$$

Also, $\frac{du}{dx}=\frac{d}{dx}\left(\frac{1+x}{1-x}\right)=\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}=\frac{1-x+1+x}{(1-x)^2}=\frac{2}{(1-x)^2}$.

Thus, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2}\cdot\sqrt{\frac{1-x}{1+x}}\cdot\frac{2}{(1-x)^2}=\frac{1}{\sqrt{1+x}\,(1-x)^{3/2}}$

8. **The General Power Rule:** A direct result of the Chain Rule is a very useful rule, called the general power rule, that is; if $y=u^n$, where u is a differentiable function of x, then $\frac{dy}{dx}=nu^{n-1}\cdot\frac{du}{dx}$

For example, if $y=\sqrt{\frac{1+x}{1-x}}$, then

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2}\cdot\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2}\cdot\frac{1-x+1+x}{(1-x)^2}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2}\cdot\frac{2}{(1-x)^2}=\frac{1}{\sqrt{1+x}\,(1-x)^{3/2}}.$$

9. The Parametric Equations and Their Derivative

If $y=f(t)$ and $x=g(t)$ are parametric equations of a curve then: $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$

For example, let $y=t^3+2t^2-1$ and $x=t^2-5t$ then:

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\left(3t^2+4t-0\right)\div(2t-5)=\frac{3t^2+4t}{2t-5}$$

Derivatives of Trigonometric Functions

In this section we shall derive the formulae for the derivatives of trigonometric functions.

- **Find the derivative of y = sin x by ab-initio method**

Let Δx be a change in x and Δy be a corresponding change in y, so that

$$y+\Delta y=\sin(x+\Delta x)\Rightarrow\Delta y=\sin(x+\Delta x)-y\Rightarrow\Delta y=\sin(x+\Delta x)-\sin x$$

Using the formula $\sin u-\sin v=2\cos\frac{u+v}{2}\sin\frac{u-v}{2}$, we get

$$\Delta y=2\cos\frac{x+\Delta x+x}{2}\sin\frac{x+\Delta x-x}{2}\Rightarrow\Delta y=2\cos\frac{2x+\Delta x}{2}\sin\frac{\Delta x}{2}$$

➔ $$\Delta y=2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x}=\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}\Rightarrow\frac{\Delta y}{\Delta x}=\frac{\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\Rightarrow\frac{\Delta y}{\Delta x}=\cos\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

Taking limit $\Delta x\to 0$, we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

Applying the limit, we get

$$\frac{dy}{dx}=\cos x\times 1=\cos x \qquad \left[\text{NOTE}: \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right].$$

Hence, if $y=\sin x$ then $\frac{dy}{dx}=\cos x$. That is : $\frac{d}{dx}(\sin x)=\cos x$

REMARK: You may find the derivatives of all trigonometric functions in any elementary book on calculus or you may seek help from your tutor to derive the following by ab-initio method.

y	dy/dx	y	dy/dx
sin x	cos x	cot x	$-\text{cosec}^2 x$
cos x	- sin x	sec x	sec x tan x
tan x	$\sec^2 x$	cosec x	- cosec x cot x

Example 01: Find the derivatives of the following functions

(i) $y=\frac{\sin x}{1+\cos x}$ (ii) $y=\tan(2x-3)$

Solution: (i) $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)=\frac{(1+\cos x)(\sin x)'-\sin x\,(1+\cos x)'}{(1+\cos x)^2}$.

NOTE: Dash " ` " stands for derivative.

$$=\frac{(1+\cos x)(\cos x)-(\sin x)(0-\sin x)}{(1+\cos x)^2}$$

➔ $\frac{dy}{dx}=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}=\frac{1+\cos x}{(1+\cos x)^2}=\frac{1}{1+\cos x}$ $\left[\cos^2 x+\sin^2 x=1\right]$

(ii) $\frac{dy}{dx}=\frac{d}{dx}\left[\tan(2x-3)\right]=\sec^2(2x-3)\frac{d}{dx}(2x-3)=\sec^2(2x-3)(2)=2\sec^2(2x-3)$

Derivatives of Inverse Trigonometric Functions

- **Find the derivative of $y=\sin^{-1}x$**

Solution: Given that $y=\sin^{-1}x$ ➔ $x=\sin y$. Differentiating both sides w. r. t y, we get:

$$\frac{dx}{dy}=\cos y=\cos y\Rightarrow\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

Hence, if $y=\sin^{-1}x$ then $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$ or $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}, x\neq 1$

REMARK: Students are advised to find the derivatives of remaining five inverse trigonometric functions. The following table shows the derivatives of all inverse trigonometric functions.

y	dy/dx	y	dy/dx
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, x\neq 1$	$\cot^{-1} x$	$\frac{-1}{1+x^2}$

$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}, x \neq 1$	$\sec^{-1} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\operatorname{cosec}^{-1} x$	$\dfrac{-1}{x\sqrt{x^2-1}}$

Example 02: Find the derivative of $y = \cot^{-1}\left(\dfrac{2x}{1-x^2}\right)$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^{-1}\left(\frac{2x}{1-x^2}\right)\right] = \frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right) \quad \left[\text{NOTE}: \frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}\right]$$

$$= \frac{-1\left(1-x^2\right)^2}{\left(1-x^2\right)^2+4x^2}\left[\frac{\left(1-x^2\right)(2)-(2x)(-2x)}{\left(1-x^2\right)^2}\right] = \frac{-\left(1-x^2\right)^2}{1-2x^2+x^4+4x^2}\left[\frac{2-x^2+4x^2}{\left(1-x^2\right)^2}\right] \rightarrow$$

$$\frac{dy}{dx} = \frac{-\left(2+2x^2\right)}{1+2x^2+x^4} = \frac{-2\left(1+x^2\right)}{(1+x^2)^2} = -\frac{1}{(1+x^2)}$$

Example 03: Differentiate $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ **with respect to** $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Solution: Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$; $z = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

It is required to find dy/dz. Put $x = \tan\theta$, we get:

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta \qquad \Rightarrow dy/d\theta = 2 \qquad (1)$$

Similarly, $z = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{\sec^2\theta}\right) = \sin^{-1}(2\sin\theta\cos\theta)$

$$z = \sin^{-1}(\sin 2\theta) = 2\theta \qquad \Rightarrow dz/d\theta = 2 \qquad (2)$$

From (1) and (2), we have: $\dfrac{dy}{dz} = \dfrac{dy}{d\theta} \div \dfrac{dz}{d\theta} = 2 \div 2 = 1$

$$\text{NOTE}: \tan(a+b) = \frac{\tan a + \tan b}{1-\tan a\tan b} \Rightarrow \tan 2\theta = \tan(\theta+\theta) = \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$$

And $\dfrac{2\tan\theta}{\sec^2\theta} = \dfrac{2\sin\theta}{\cos\theta}.\cos^2\theta = 2\sin\theta\cos\theta = \sin 2\theta$

Example 04: Differentiate $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ **with respect to** $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Solution: Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $z = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

It is required to find dy/dz. Putting $x = \tan\theta$, we get,

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta \qquad \Rightarrow \frac{dy}{d\theta} = 2 \qquad (1)$$

Similarly, $z=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}\left[\frac{\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)}{\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)}\right]=\cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$

$z=\cos^{-1}(\cos 2\theta)=2\theta \qquad \Rightarrow \frac{dz}{d\theta}=2$ (2)

From (1) and (2), $\frac{dy}{dz}=\frac{dy}{d\theta}\div\frac{dz}{d\theta}=2\div 2=1$

Example 05: Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ **with respect to** $\tan^{-1}x$

Solution: Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $z=\tan^{-1}x$

It is required to find dy/dz. Putting $x=\tan\theta$, we get,

$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$=\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2.\cos\theta/2}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2} \qquad \Rightarrow \frac{dy}{d\theta}=\frac{1}{2}$ (1)

Similarly, $z=\tan^{-1}(\tan\theta)=\theta \qquad \Rightarrow \frac{dz}{d\theta}=1$ (2)

From (1) and (2), $\frac{dy}{dz}=\frac{dy}{d\theta}\div\frac{dz}{d\theta}=\frac{1}{2}\div 1=\frac{1}{2}$

Example 06: Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ **with respect to** $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

Solution: Let $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $z=\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

It is required to find dy/dz. Putting $x=\sin\theta$, we get,

$y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2}}\right)=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\tan^{-1}(\tan\theta)=\theta$

$\Rightarrow \qquad dy/d\theta=1$ (1)

Similarly, $z=\sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right)=\sec^{-1}\left[\frac{1}{\cos 2\theta}\right]=\sec^{-1}(\sec 2\theta)=2\theta$

$\Rightarrow \qquad dz/d\theta=2$ (2)

From (1) and (2), $\frac{dy}{dz}=\frac{dy}{d\theta}\div\frac{dz}{d\theta}=1\div 2=\frac{1}{2}$

REMAK: See the change in substitution in this example and examples 3, 4, and 5.

Derivatives of Exponential and Logarithmic Functions

In this section we shall find the derivatives of:

(1) $\log_a x$ **(2)** $\log_e x=\log x$ **(3)** $a^x, a>0$ **(4)** e^x

1. If $y=\log_a x,\ (x>0, a>1)$ then $\frac{dy}{dx}=\frac{1}{x}\log_a e=\frac{1}{x\ln a}$

Proof: $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}=\lim_{h\to 0}\frac{\log_a\left(\frac{x+h}{x}\right)}{h}$

$$=\lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)=\lim_{h\to 0}\frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)=\frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$\frac{dy}{dx}=\frac{1}{x}\log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]=\frac{1}{x}\log_a e=\frac{1}{x\log_e a}=\frac{1}{x\ln a}$$

REMARKS: (i) $\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=e$ **(ii)** $\log_a b=\frac{\log_e b}{\log_e a}=\frac{\ln a}{\ln b}$

➔ $$\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e=\frac{1}{x\ln a}$$

2. If we put $a=e$ in (1) then $y=\log_e x\Rightarrow y=\ln x$, so that

$\frac{dy}{dx}=\frac{d}{dx}(\ln x)=\frac{1}{x\ln e}=\frac{1}{x}\cdot\ [\ln e=1]$ ➔ $\frac{d}{dx}(\ln x)=\frac{1}{x}\log_e e=\frac{1}{x\ln e}=\frac{1}{x}$

3. If $y=a^x$, then $\frac{dy}{dx}=a^x\ln a$

Proof: We have $f(x)=y=a^x$ By definition,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$\frac{dy}{dx}=\lim_{h\to 0}\left[a^x\left(\frac{a^h-1}{h}\right)\right]=a^x\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)=a^x\ln a\qquad\left[\text{Note: }\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)=\ln a\right]$$

➔ $$y=a^x,\text{ then }\frac{dy}{dx}=a^x\ln a$$

4. If $f(x)=y=e^x$, then $\frac{dy}{dx}=e^x$

Proof: Using the result of (3) if we put a = e, we obtain

$$\frac{d}{dx}(e^x)=e^x\cdot\ln e=e^x$$

Alternative Methods:

(i) Let $y=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ This is the series expansion of e^x.

Differentiating both sides w . r . t x, we get,

$$\frac{dy}{dx}=0+1+\frac{2x}{2!}+\frac{3x^3}{3.2!}+\frac{4x^3}{4.3!}+\ldots=1+x+\frac{x^3}{2!}+\frac{x^3}{3!}+\ldots=e^x$$

Thus if y= e^x then dy/dx = e^x.

(ii) $y=\ln x=\log_e x$. Taking the antilog on both sides, we get: $e^y=x$.

Differentiating both sides w. r. t y, we get

$\frac{dx}{dy} = e^y \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$. Thus if y = ln x then, dy/dx = 1/x.

(iii) Let $y = a^x$. Taking log on both sides, we get: $\ln y = \ln a^x \rightarrow \ln y = x \ln a$
$\rightarrow x = \ln y / \ln a$.

Differentiate both sides w . r . t y, we get

$\frac{dx}{dy} = \frac{1}{y \ln a} \Rightarrow \frac{dy}{dx} = y \ln a = a^x \ln a$. Thus if $y = a^x$ then, $dy/dx = a^x \ln a$.

(iv) Let $y = \log_a x$. Taking the antilog on both sides, we get $a^y = x$. Differentiating both sides w . r . t y, we get

$a^x \ln a = \frac{dx}{dy} \Rightarrow \frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$. Thus if $y = \log_a x \Rightarrow \frac{dy}{dx} = \frac{1}{x \ln a}$

The following table summarizes the above formulae.

Y	dy/dx	y	dy/dx
e^x	e^x	$\ln x$	$1/x$
a^x	$a^x \ln x$	$\log_a x$	$1/x \ln a$

Example 07: Find the derivatives of the following functions.

(i) $y = \ln\left(\frac{1-x^2}{1+x^2}\right)$ (ii) $y = 3^{2x^2+x}$ (iii) $y = e^{2x^2+x}$

Solution:

(i) $\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\frac{1-x^2}{1+x^2}\right)\right] = \frac{1}{\left(\frac{1-x^2}{1+x^2}\right)} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$, $\left[\text{NOTE}: \frac{d}{dx}(\ln x) = \frac{1}{x}\right]$

$\rightarrow \frac{dy}{dx} = \left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}\right] = \frac{-2x-2x^3-2x+2x^3}{(1-x^2)(1+x^2)} = \frac{-4x}{1-x^4}$

(ii) $\frac{dy}{dx} = \frac{d}{dx}\left(3^{2x^2+x}\right) = 3^{2x^2+x} \cdot \frac{d}{dx}(2x^2+x) \cdot \ln 3$, using $\frac{d}{dx}(a^x) = a^x \cdot \ln a$

$= 3^{2x^2+x}(4x+1)\ln 3 = (4x+1)3^{2x^2+x}\ln 3$

(iii) $\frac{dy}{dx} = \frac{d}{dx}\left(e^{2x^2+x}\right) = e^{2x^2+x} \cdot \frac{d}{dx}(2x^2+x)$, using $\frac{d}{dx}(e^x) = e^x$

$= e^{2x^2+x}(4x+1) = (4x+1)e^{2x^2+x}$

Derivatives of Hyperbolic Functions

We know that hyperbolic functions are defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \text{sech} x = \frac{2}{e^x + e^{-x}}, \text{csch} x = \frac{2}{e^x - e^{-x}}$$

Consider,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \frac{(e^x + e^{-x})}{2} = \cosh x$$

Therefore, $\frac{d}{dx}(\sinh x) = \cosh x$

Similarly, the derivatives of the other hyperbolic functions can be found. The complete list is given in the following table. Students are advised to find these derivatives seeking help of their tutors.

y	dy/dx	y	dy/dx
sinh x	cosh x	coth x	$-\text{cosech}^2 x$
cosh x	sinh x	sech x	-sech x tanh x
tanh x	$\text{sech}^2 x$	cosech x	- cosech x coth x

Derivatives of Inverse Hyperbolic Functions

Inverse hyperbolic functions correspond to inverse circular functions and their derivatives are found by similar methods.

1. Show that: $\frac{d}{dx}\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}}$

Let $y=\sinh^{-1} x \Rightarrow x=\sinh y$

Differentiating with respect to y, we get

$$\frac{dx}{dy}=\cosh y \Rightarrow \frac{dy}{dx}=\frac{1}{\cosh y}=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

Hence, $\frac{d}{dx}\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}}$

The derivatives of other inverse hyperbolic functions may be found in a similar way. The derivatives are produced in the following table. The readers are advised to solve them taking the help of their teachers.

y	dy/dx	y	dy/dx
$\sinh^{-1} x$	$1/\sqrt{x^2+1}$	$\coth^{-1} x$	$1/(1-x^2)$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$	$\text{Sech}^{-1} x$	$-1/x\sqrt{1-x^2}$
$\tanh^{-1} x$	$1/(1-x^2)$	$\text{Cosec}^{-1} x$	$-1/x\sqrt{1+x^2}$

Example 08: Differentiate the following functions

(i) $f(x)=xa^x \sinh x$ (ii) $f(x)=\sinh^{-1}(\tanh x)$

Solution: (i) We have $f(x)=xa^x \sinh x$, differentiating, we get

$$\frac{d}{dx}[f(x)]=\frac{d}{dx}\left(xa^x \sinh x\right)=xa^x\frac{d}{dx}(\sinh x)+x\frac{d}{dx}\left(a^x\right)\sinh x+\frac{d}{dx}(x)a^x \sinh x$$

$$=xa^x\cosh x+xa^x(\ln a)\sinh x+a^x\sinh x$$

(ii) $$\frac{d}{dx}[f(x)]=\frac{d}{dx}\left[\sinh^{-1}(\tanh x)\right]=\frac{1}{\sqrt{\tanh^2 x+1}}\frac{d}{dx}(\tanh x)$$

$$=\frac{\text{sech}^2 x}{\sqrt{\tanh^2 x+1}} \quad \left[\text{since } \frac{d}{dx}\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}} \text{ and } \frac{d}{dx}(\tanh x)=\text{sech}^2 x\right]$$

Logarithmic Differentiation

If $f(x)=u^v$, where both u and v are functions of x, the derivative of f(x) can be obtained by taking natural logarithm on both sides of given equation and then differentiating according to rules of derivatives. This process of finding the derivative in such cases is called "logarithmic differentiation."

Example 09: Find the derivative of the function: $y = x^{\cos x}$

Solution: Given $y = x^{\cos x}$, taking log on both sides, we get

$$\ln y = \ln\left(x^{\cos x}\right) \Rightarrow \ln y = \cos x.\ln x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{x}\right) + \ln x(-\sin x) = \frac{\cos x}{x} - \sin x \ln x$$

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

Example 10: Differentiate $(\sin x)^x$ with respect to $(\cos x)^x$.

Solution: Let $u = (\sin x)^x$ and $v = (\cos x)^x$. It is required to find du/dv.

First we take, $u = (\sin x)^x$. Taking ln, we get: $\ln u = x\ln(\sin x)$.

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1) \Rightarrow \frac{du}{dx} = u\left[x\cot x + \ln(\sin x)\right]$$

$$\frac{du}{dx} = (\sin x)^x\left[x\cot x + \ln(\sin x)\right] \qquad (1)$$

Now let, $v = (\cos x)^x$. Taking log, we get: $\ln v = x\ln(\cos x)$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = x\left(\frac{1}{\cos x}\right)(-\sin x) + \ln(\cos x)(1) \Rightarrow \frac{dv}{dx} = v\left[-x\tan x + \ln(\cos x)\right]$$

$$\frac{dv}{dx} = (\cos x)^x\left[-x\tan x + \ln(\cos x)\right] \qquad (2)$$

Since $\frac{du}{dv} = \frac{du}{dx}\Big/\frac{dv}{dx}$, then from (1) and (2), we have

$$\frac{du}{dv} = \frac{(\sin x)^x\left[x\cot x + \ln(\sin x)\right]}{(\cos x)^x\left[-x\tan x + \ln(\cos x)\right]}$$

Implicit Differentiation

In the previous sections we have learnt the derivatives of explicit functions $y = f(x)$. In practice, functions do not always occur in explicit form. Functions which are not explicit are known as implicit. They are of the form $f(x, y) = 0$. For example, the equations:

$$x^2 + y^2 = 4, \qquad y^3 - x^3 + xy = 3, \qquad \sin y + e^x - \ln y = 0$$

provide relationships between x and y in which y is not expressed explicitly in terms of x.

Let us consider the first equation which gives on simplification: $y = \pm\sqrt{4 - x^2}$. We observe that for one value of x, there correspond two values of y. Thus, this equation does not represent a function. In fact implicit functions are not functions. They are called function because they involve relationship between x and y. The process used to determine the derivative in such cases is called **"Implicit Differentiation."**

The following procedure will be employed to determine dy/dx by implicit differentiation.

- Differentiate both sides of equation with respect to x.
- Collect all terms containing dy/dx on one side and all other terms on the other side.
- Factor out dy/dx from all terms that contain it.

- Solve for dy/dx after dividing both sides by the coefficient of dy/dx.

Example 11: Differentiate the following equation with respect to x

$$y\sin^{-1}x - x\tan^{-1}y = 1$$

Solution: Given, $y\sin^{-1}x - x\tan^{-1}y = 1$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\left(y\sin^{-1}x\right)-\frac{d}{dx}\left(x\tan^{-1}y\right)=\frac{d}{dx}(1)$$

$$\rightarrow \left[y\frac{d}{dx}\left(\sin^{-1}x\right)+\sin^{-1}x\frac{d}{dx}(y)\right]-\left[x\frac{d}{dx}\tan^{-1}y+\tan^{-1}y\frac{d}{dx}(x)\right]=0$$

$$\rightarrow \left[y\left(\frac{1}{\sqrt{1-x^2}}\right)+\sin^{-1}x\left(\frac{dy}{dx}\right)\right]-\left[x\left(\frac{1}{1+y^2}\right)\left(\frac{dy}{dx}\right)+\tan^{-1}y(1)\right]=0$$

$$\rightarrow \frac{y}{\sqrt{1-x^2}}+\sin^{-1}x\left(\frac{dy}{dx}\right)-\frac{x}{1+y^2}\left(\frac{dy}{dx}\right)-\tan^{-1}y=0$$

$$\rightarrow \left(\sin^{-1}x-\frac{x}{1+y^2}\right)\left(\frac{dy}{dx}\right)=\tan^{-1}y-\frac{y}{\sqrt{1-x^2}}$$

$$\rightarrow \left[\frac{(1+y^2)\sin^{-1}x-x}{1+y^2}\right]\left(\frac{dy}{dx}\right)=\frac{\sqrt{1-x^2}\tan^{-1}y-y}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx}=\frac{(1+y^2)\left[\sqrt{1-x^2}\tan^{-1}y-y\right]}{\sqrt{1-x^2}\left[(1+y^2)\sin^{-1}x-x\right]}$

Example 12: In each of the following find dy/dx

(i) $y=\sqrt{a^2+x^2}$

Solution: Given $y=\sqrt{a^2+x^2}$. Differentiating w.r.t x, we get

$$y'=\frac{1}{2}\left(a^2+x^2\right)^{-1/2}\cdot\frac{d}{dx}\left(a^2+x^2\right)=\frac{1}{2}\frac{2x}{\sqrt{a^2+x^2}}=\frac{x}{\sqrt{a^2+x^2}}$$

(ii) $y=\sqrt[3]{x^3+x+1}$

Solution: Given that $y=\sqrt[3]{x^3+x+1}=\left(x^3+x+1\right)^{1/3}$. Differentiating w.r.t x

$$y'=\frac{1}{3}\left(x^3+x+1\right)^{-2/3}\frac{d}{dx}\left(x^3+x+1\right)=\frac{3x^2+1}{3\left(x^3+x+1\right)^{2/3}}$$

(iii) $y=\frac{\sqrt{\sin x}}{\sin\sqrt{x}}$

Solution: Given that $y=\frac{\sqrt{\sin x}}{\sin\sqrt{x}}$. Differentiating w.r.t x, we get

$$y' = \frac{\sin\sqrt{x}\,\frac{d}{dx}\sqrt{\sin x} - \sqrt{\sin x}\,\frac{d}{dx}\sin\sqrt{x}}{\left(\sin\sqrt{x}\right)^2}$$

$$= \frac{\sin\sqrt{x}.\frac{1}{2}(\sin x)^{-1/2}\cos x - \sqrt{\sin x}\cos\sqrt{x}.\frac{1}{2}(x)^{-1/2}}{\left(\sin\sqrt{x}\right)^2}$$

$$= \frac{\frac{\sin\sqrt{x}}{2\sqrt{\sin x}}\cos x - \frac{\sqrt{\sin x}\cos\sqrt{x}}{2\sqrt{x}}}{\left(\sin\sqrt{x}\right)^2} = \frac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}\sin^2\sqrt{x}}$$

(iv) $y = \sqrt{\log_{10}\left(x^2+1\right)}$

Solution: Given that $y = \sqrt{\log_{10}\left(x^2+1\right)} = \left[\log_{10}\left(x^2+1\right)\right]^{1/2}$. Differentiating w.r.t x, we get

$$y' = \frac{1}{2}\left[\log_{10}\left(x^2+1\right)\right]^{-1/2}.\frac{d}{dx}\log_{10}\left(x^2+1\right)$$

$$= \frac{1}{2\sqrt{\log_{10}\left(x^2+1\right)}}.\frac{1}{\left(x^2+1\right).\ln 10}.\frac{d}{dx}\left(x^2+1\right) = \frac{2x}{2\ln 10\left(x^2+1\right)\sqrt{\log_{10}\left(x^2+1\right)}}$$

$$= \frac{x}{\left(x^2+1\right)\sqrt{\log_{10}\left(x^2+1\right)}.\ln 10}$$

(v) $y = \tan(\sin x)$

Solution: Given $y = \tan(\sin x)$. Differentiate w.r.t x, we get

$$y' = \sec^2(\sin x).\frac{d}{dx}\sin x = \cos x\ \sec^2(\sin x)$$

(vi) $y = \tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)$, α being constant.

Solution: Let $z = \frac{x\sin\alpha}{1-x\cos\alpha} \Rightarrow y = \tan^{-1} z$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{1+z^2}.\left[\frac{(1-x\cos\alpha).\sin\alpha - x\sin\alpha(0-\cos\alpha)}{(1-x\cos\alpha)^2}\right] \qquad (1)$$

Consider

$$\frac{1}{1+z^2} = \frac{1}{1+\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)^2} = \frac{(1-x\cos\alpha)^2}{(1-x\cos\alpha)^2+(x\sin\alpha)^2}$$

$$= \frac{(1-x\cos\alpha)^2}{\left(1-2x\cos\alpha+x^2\cos^2\alpha+x^2\sin^2\alpha\right)} = \frac{(1-x\cos\alpha)^2}{\left(1-2x\cos\alpha+x^2\right)}. \qquad \left[\cos^2\alpha+\sin^2\alpha=1\right]$$

Thus equation (1) becomes:

$$\frac{dy}{dx}=\frac{(1-x\cos\alpha)^2}{\left(1-2x\cos\alpha+x^2\right)}\left(\frac{1-x\sin\alpha\cos\alpha+x\sin\alpha\cos\alpha}{(1-x\cos\alpha)^2\,.}\right)=\frac{1}{\left(1-2x\cos\alpha+x^2\right)}$$

(vii) $y=\ln\left(\frac{x^2+x+1}{x^2x+1}\right)$

Solution: Given $y=\ln\left(\frac{x^2+x+1}{x^2x+1}\right)=\ln\left(x^2+x+1\right)-\ln\left(x^2-x+1\right)$

Differentiating w.r.t x, we get

$$y'=\frac{1}{x^2+x+1}\frac{d}{dx}\left(x^2+x+1\right)-\frac{1}{x^2-x+1}\frac{d}{dx}\left(x^2-x+1\right)=\frac{2x+1}{\left(x^2+x+1\right)}-\frac{2x-1}{\left(x^2-x+1\right)}$$

Simplifying, we get: $y'=\frac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}=\frac{2\left(1-x^2\right)}{\left(x^4+x^2+1\right)}$

(viii) $y=x^{x^2}$

Solution: Given $y=x^{x^2}$. Taking ln on both sides, we get

$\ln y=\ln\left(x^{x^2}\right)=x^2\ln x$. Differentiating w.r.t x, we get

$$\frac{1}{y}y'=x^2\frac{d}{dx}\ln x+\ln x\frac{d}{dx}x^2=x^2.\frac{1}{x}+\ln x.(2x)=x+2x\ln x=x(1+2\ln x)$$

Thus, $y'=yx(1+2\ln x)=x.x^{x^2}(1+2\ln x)=x^{x^2+1}(1+2\ln x)$

(ix) $y=\left(\sin^{-1}x\right)^{x^{1/x}}$

Solution: Given that $y=\left(\sin^{-1}x\right)^{x^{1/x}}$. Taking ln on both sides, we get

$\ln y=\ln\left(\sin^{-1}x\right)^{x^{1/x}}=x^{1/x}.\ln\left(\sin^{-1}x\right)$. Differentiating w.r. t x, we get:

$$\frac{1}{y}y'=x^{1/x}\frac{d}{dx}\ln\left(\sin^{-1}x\right)+\ln\left(\sin^{-1}x\right)\frac{d}{dx}x^{1/x} \qquad (1)$$

Now $\frac{d}{dx}\ln\left(\sin^{-1}x\right)=\frac{1}{\sin^{-1}x}.\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}\sin^{-1}x}$

To find $\frac{d}{dx}x^{1/x}$, we let $z=x^{1/x}$. Taking log, we get

$\ln z=\frac{1}{x}\ln x=\frac{\ln x}{x}$. Differentiating w.r.t x, we get:

$$\frac{1}{z}\frac{d}{dx}z=\frac{x.(1/x)-\ln x.1}{x^2}\quad\Rightarrow z'=\frac{d}{dx}\left(x^{1/x}\right)=z\frac{1-\ln x}{x^2}=x^{1/x}\frac{1-\ln x}{x^2}$$

Thus equation (1) becomes:

$$\frac{1}{y}y'=x^{1/x}\frac{d}{dx}\ln\left(\sin^{-1}x\right)+\ln\left(\sin^{-1}x\right)\frac{d}{dx}x^{1/x}$$

$$= x^{1/x}\frac{1}{\sqrt{1-x^2}\sin^{-1}x} + \ln\left(\sin^{-1}x\right)x^{1/x}\frac{1-\ln x}{x^2}$$

$$\Rightarrow y' = y\left[x^{1/x}\frac{1}{\sqrt{1-x^2}\sin^{-1}x} + \ln\left(\sin^{-1}x\right)x^{1/x}\frac{1-\ln x}{x^2}\right]$$

Or $y' = x^{1/x}\left(\sin^{-1}x\right)^{x^{1/x}}\left[\frac{1}{\sqrt{1-x^2}\sin^{-1}x} + \frac{\ln\left(\sin^{-1}x\right)(1-\ln x)}{x^2}\right]$

(x) $x^y = e^{x-y}$

Solution: Taking ln on both sides, we get:

$\ln(x^y) = \ln(e^{x-y}) \Rightarrow y.\ln x = (x-y)\ln e = x - y$ [NOTE: ln e = 1]

Thus, $y(1+\ln x) = x \quad \Rightarrow y = \frac{x}{1+\ln x}$.

Now differentiating both sides w.r.t x, we get:

$$y' = \frac{(1+\ln x).\frac{d}{dx}x - x.\frac{d}{dx}(1+\ln x)}{(1+\ln x)^2} = \frac{(1+\ln x).1 - x.\frac{1}{x}}{(1+\ln x)^2} = \frac{1+\ln x - 1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

(xi) $y^x + x^y = c$

Solution: Let $u = y^x$ and $v = x^y$. Then given equation becomes: $u + v = c$

Differentiating both sides w.r.t x, we get: $\frac{du}{dx} + \frac{dv}{dx} = 0$ (1)

Now: $u = y^x$	:	$v = x^y$
Taking ln on both sides	:	Taking ln on both sides
$\ln u = \ln y^x$	:	$\ln v = \ln x^y$
$= x\ln y$	:	$= y.\ln x$
Differentiating:	:	Differentiating:
$\frac{1}{u}.\frac{du}{dx} = x\frac{d}{dx}\ln y + \ln y.\frac{dx}{dx}$	:	$\frac{1}{v}.\frac{dv}{dx} = \ln x\frac{d}{dx}y + y.\frac{d}{dx}\ln x$
$\frac{du}{dx} = u\left[x\frac{y'}{y} + \ln y.1\right]$	:	$\frac{dv}{dx} = v\left[\ln x.y' + y.\frac{1}{x}\right]$

Substituting the values of u and v, we get:

$$\frac{du}{dx} = y^x\left[\frac{xy'}{y} + \ln y\right] = xy^{x-1}y' + y^x\ln y \text{ and } \frac{dv}{dx} = x^y\left[\ln x.y' + \frac{y}{x}\right] = x^y\ln x.y' + yx^{y-1}$$

Thus equation (1) becomes:

$$xy^{x-1}y' + y^x\ln y + x^y\ln x.y' + yx^{y-1} = 0 \quad \Rightarrow \left(xy^{x-1} + x^y\ln x\right)y' + \left(yx^{y-1} + y^x\ln y\right) = 0$$

$$\Rightarrow \quad y' = -\left[\left(yx^{y-1} + y^x\ln y\right)/\left(xy^{x-1} + x^y\ln x\right)\right]$$

(xii) $\left[(x+y)/(x-y)\right] = x^2 + y^2$

Solution: Simplifying given equation, we get: $(x + y) = (x - y)(x^2 + y^2)$

$\Rightarrow x + y = x^3 - x^2y + xy^2 - y^3 \quad \Rightarrow x^3 - y^3 - x^2y + xy^2 - x - y = 0$

Differentiating both sides w.r.t x, we get

$$\frac{d}{dx}x^3 - \frac{d}{dx}y^3 - \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(xy^2\right) - \frac{d}{dx}x - \frac{d}{dx}y = \frac{d}{dx}0$$

$$\Rightarrow 3x^2 - 3y^2y' - (x^2y' + 2xy) + (2xyy' + y^2) - 1 - y' = 0$$

Or $\left(2xy - x^2 - 3y^2 - 1\right)y' - \left(2xy - 3x^2 - y^2 + 1\right) = 0$

$$\Rightarrow \quad y' = \left(2xy - 3x^2 - y^2 + 1\right)/\left(2xy - x^2 - 3y^2 - 1\right)$$

(xiii) $x + \sin^{-1} y = xy$

Solution: Given equation is: $x + \sin^{-1} y - xy = 0$. Differentiating w.r.t x, we get

$$\frac{d}{dx}x + \frac{d}{dx}\sin^{-1} y - \frac{d}{dx}(xy) = \frac{d}{dx}(0) \quad \Rightarrow \quad 1 + \frac{1}{\sqrt{1-y^2}}y' - xy' - y.1 = 0$$

$$\Rightarrow \left[\frac{1}{\sqrt{1-y^2}} - x\right]y' = y - 1 \Rightarrow \left[\frac{1 - x\sqrt{1-y^2}}{\sqrt{1-y^2}}\right]y' = y - 1 \quad \Rightarrow y' = \frac{(y-1)\sqrt{1-y^2}}{1 - x\sqrt{1-y^2}}$$

(viv) $y = \ln\left(e^x / \left(1 + e^x\right)\right)$

Solution: Given that

$$y = \ln\left(\frac{e^x}{1+e^x}\right) = \ln\left(e^x\right) - \ln\left(1+e^x\right) = x\ln e - \ln\left(1+e^x\right) = x - \ln\left(1+e^x\right) \quad [\ln e = 1]$$

Differentiating w.r.t x, we get

$$y' = 1 - \frac{1}{1+e^x}\frac{d}{dx}\left(1+e^x\right) = 1 - \frac{e^x}{1+e^x} = \frac{1+e^x-e^x}{1+e^x} = \frac{1}{1+e^x}$$

(xv) $y = x^{\ln x}$

Solution: Given that $y = x^{\ln x}$. Taking ln on both sides, we obtain

$\ln y = \ln\left(x^{\ln x}\right) = \ln x.\ln x = (\ln x)^2$. Differentiating, we get:

$$\frac{1}{y}y' = 2(\ln x).\frac{d}{dx}\ln x \quad \Rightarrow \quad y' = 2y\ln x.\frac{1}{x} = \frac{2\ln x\, x^{\ln x}}{x}$$

(xvi) $y = x.a^x.\sinh x$

Solution: Given that: $y = x.a^x.\sinh x$. Differentiate w.r.t x, we get

$y' = xa^x(\sinh x)' + x\sinh x\left(a^x\right)' + a^x\sinh x(x)'$, where $(\ )'$ indicates derivative.

$$= xa^x\cosh x + x\sinh x\ a^x\ln a + a^x\sinh x.1 = a^x\left[x\cosh x + x\sinh x\ln a + \sinh x\right]$$

(xvii) $y = \sec^{-1}(\sinh x)$

Solution: Differentiating both sides w.r.t x, we get

$$y' = \frac{1}{\sinh x\sqrt{\sinh^2 x - 1}}.\frac{d}{dx}(\sinh x) = \frac{1}{\sinh x}\frac{\cosh x}{\sqrt{\sinh^2 x - 1}} = \frac{\coth x}{\sqrt{\sinh^2 x - 1}}$$

$$\left[\text{NOTE}: \left(\sec^{-1} x\right)' = \frac{1}{x\sqrt{x^2 - 1}}\right]$$

(xviii) $x = 3at/(1 + t^2)$, $y = 3at^2/(1 + t^2)$

Solution: Given equations are parametric equations, hence $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ (1)

$$\text{Now } \frac{dy}{dt} = \frac{\left(1+t^2\right)\left(3at^2\right)' - \left(3at^2\right)\left(1+t^2\right)'}{\left(1+t^2\right)^2} = \frac{\left(1+t^2\right)(6at) - 3at^2(0+2t)}{\left(1+t^2\right)^2} = \frac{6at}{\left(1+t^2\right)^2}$$

And $\frac{dy}{dt}=\frac{(1+t^2)(3at)'-(3at)(1+t^2)'}{(1+t^2)^2}=\frac{(1+t^2)(3a)-3at(0+2t)}{(1+t^2)^2}=\frac{3a(1-t^2)}{(1+t^2)^2}$

Thus equation (1) will become:

$$\frac{dy}{dx}=\frac{6at}{(1+t^2)^2}\div\frac{3a(1-t^2)}{(1+t^2)^2}=\frac{6at}{(1+t^2)^2}\times\frac{(1+t^2)^2}{3a(1-t^2)}=\frac{2t}{(1-t^2)}$$

(xix) Differentiate logarithmically when $y=\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}$

Solution: Given $y=\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}=\left[\frac{x(x^2+1)}{(x-1)^2}\right]^{1/3}$

Taking ln on both sides, we get: $\ln y=\frac{1}{3}\left[\ln x+\ln(x^2+1)-2\ln(x-1)\right]$

Differentiating both sides, we get:

$$\frac{1}{y}y'=\frac{1}{3}\left[\frac{1}{x}+\frac{2x}{(x^2+1)}-\frac{2}{(x-1)}\right]\Rightarrow y'=\frac{y}{3}\left[\frac{1}{x}+\frac{2x}{(x^2+1)}-\frac{2}{(x-1)}\right]$$

Substituting the value of y, we get:

$$y'=\frac{1}{3}\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}\left[\frac{1}{x}+\frac{2x}{(x^2+1)}-\frac{2}{(x-1)}\right]$$

(xx) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$

Solution: Let $u=(\tan x)^{\cot x}$ and $v=(\cot x)^{\tan x}$. Then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad (1)$$

Now $u=(\tan x)^{\cot x}$. Taking ln on both sides, we get

$$\ln u=\ln(\tan x)^{\cot x}=\cot x.\ln(\tan x)$$

Differentiating, we obtain:

$$\frac{1}{u}u'=\cot x\frac{d}{dx}\ln(\tan x)+\ln(\tan x)\frac{d}{dx}(\cot x)=\frac{\cot x.\sec^2 x}{\tan x}+\ln(\tan x)(-\operatorname{cosec}^2 x)$$

➔ $u'=u\left[\cot^2 x\sec^2 x-\operatorname{cosec}^2 x\ln(\tan x)\right]$

Now consider, $v=(\cot x)^{\tan x}$. Taking log on both sides, we get

$$\ln v=\ln(\cot x)^{\tan x}=\tan x.\ln(\cot x)$$

Differentiating, we obtain:

$$\frac{1}{v}v'=\tan x\frac{d}{dx}\ln(\cot x)+\ln(\cot x)\frac{d}{dx}(\tan x)=\frac{\tan x(-\operatorname{cosec}^2 x)}{\cot x}+\ln(\cot x)(\sec^2 x)$$

$$v' = v\left[-\tan^2 x\ \text{cosec}^2 x + \sec^2 x \ln(\cot x)\right]$$

Thus equation (1) becomes:

$$y' = u\left[\cot^2 x \sec^2 x - \text{cosec}^2 x \ln(\tan x)\right] + v\left[-\tan^2 x\ \text{cosec}^2 x + \sec^2 x \ln(\cot x)\right]$$

Putting $u = (\tan x)^{\cot x}$ and $v = (\cot x)^{\tan x}$, we get:

$$y' = (\tan x)^{\cot s}\left[\cot^2 x \sec^2 x - \text{cosec}^2 x \ln(\tan x)\right]$$

$$+ (\cot x)^{\tan x}\left[-\tan^2 x\ \text{cosec}^2 x + \sec^2 x \ln(\cot x)\right]$$

(xxi) $y = x^{x^{x^{x\cdots}}}$

Solution: Given that $y = x^{x^{x^{x\cdots}}}$ ➔ $y = x^y$. (See the trick).

Taking ln on both sides, we get

$\ln y = \ln(x^y) = y.\ \ln x$. Now differentiating both sides, we get:

$$\frac{1}{y}y' = y\frac{1}{x} + y'\ln x \Rightarrow \frac{y'}{y} - y'\ln x = \frac{y}{x} \Rightarrow \frac{(1 - y\ln x)}{y}y' = \frac{y}{x} \Rightarrow y' = \frac{y^2}{x(1 - y\ln x)}$$

3.4 CASE STUDY PROBLEMS

In this section, we shall study case study problems taken from diverse areas of physical, social sciences, economics etc.

Rates of Change

Applications that follow from interpreting the derivatives as a rate of change are presented in this section. To set the stage for the rate of change interpretation, we present here examples on **average rate of change** and instantaneous rate of change.

Example 01: (AVERAGE SPEED) If Ali walks 7 miles in 2 hours, what is his average speed?

Solution: Average speed is defined as the change in distance Δs divided by the change in time Δt. Thus

Average speed $= \Delta s / \Delta t$

In this instance, $\Delta s / \Delta t$ = (7 miles)/(2 hours) =3.5miles per hours.

Hence, Ali's average speed is 3.5miles per hours.

Example 02: (AVERAGE CHANGE IN AMTRAK'S REVENUE) From 1986 to1991, M/s Dunhill's annual revenue increased from \$861,000,000 to \$1,359,000,000. What was the average change in revenue per year during this time?

Solution: The average change in revenue per year is the change in revenue ΔR divided by the change in time Δt. Thus,

$$\frac{\Delta R}{\Delta t} = \frac{1,359,000,000 - 861,000,000}{1991 - 1986} = \frac{\$498,000,000}{5\text{ years}} = \$99,600,000 \text{ per year.}$$

This shows that from 1986 to 1991, Dunhill's annual revenue increased at the average rate of \$99,600,000 per year.

Example 03: What is the average rate of change of $y = x^2$ from $x = 1$ to $x = 5$?

Solution: The average rate of change is the change in y divided by the change in x, namely $\frac{\Delta y}{\Delta x} = \frac{(5)^2 - (1)^2}{5 - 1} = \frac{25 - 1}{4} = \frac{24}{4} = 6.$

Example 04: (VELOCITY OF A FALLING OBJECT) A ball dropped from the top of a cliff will fall such that the distance it has traveled after t seconds is $s(t) = -16t^2$. What is the average velocity for the first 3 seconds? (b) How fast is the ball traveling at 3 seconds?

Solution: (a) The average velocity for the first 3 seconds is

$$\frac{\Delta s}{\Delta t}=\frac{s(3)-s(0)}{3-0}=\frac{\left[-16(3)^2\right]-\left[-16(0)^2\right]}{3}=\frac{-144}{3}=-48.$$

Thus, the average velocity is 48 feet per second. The minus sign indicates that the ball is traveling in the downward direction.

(b) The velocity at 3 seconds is an instantaneous velocity. It is the velocity at a specific time, when t=3. So we need to evaluate ds/dt at t =3. Now $s = -16t^2$:

➔ $$v=\frac{ds}{dt}=\frac{d}{dt}\left(-16t^2\right)=-16\frac{d}{dt}\left(t^2\right)=-32t \text{ feet/sec.}$$

At t =3, $$\frac{ds}{dt}=-32(3)=-96 \text{ feet/sec.}$$

This shows that the ball is traveling 96 feet per second (downward) after 3 seconds.

Example 05: Let $y = 5x^3 - x^2 + 8x + 1$. Determine the value of derivative of y when x = 4.

Solution: Given that $y = 5x^3 - x^2 + 8x + 1$. Differentiate w.r.t x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(5x^3-x^2+8x+1\right)=15x^2-2x+8.$$

Specifically, when x = 4, $\frac{dy}{dx}=15(4)^2-2(4)+8=240.$

Example 06: (POPULATION DECLINE) In 1980 the population of buffaloes in a small town, was 355,000. In 1990 it was 328,000. What is the average rate of decrease in the population per year between 1980 and 1990?

Solution: The average rate of change of decrease in the population is the change in the population ΔP divided by the change in years Δt, namely

$$\Delta P/\Delta t=(328,000-355,000)/1990-1980=-2700.$$

It means that per year 2700 buffaloes are decreased.

Example 07: (CORPORATE PROFIT) In 6 years a corporation's annual profit increased from $10,000 to $130,600. What was the average rate of increase in profit per year during this 6-year period?

Solution: Let P be the annual profit of the corporation. The average rate of increase in profit per year during 6-year period will be:

$$\frac{\Delta P}{\Delta t}=\frac{130,600-10,000}{6}=\frac{120,600}{6}=20100.$$

Thus average profit per year is $20100.

Example 08: (DIVING HAWK) Assume that a hawk dives from a height of 300 feet and that its distance from the ground at t seconds is $s=300-16t^2$ feet. What is the hawk's average velocity during the first 4 seconds?

Solution: The hawk's average velocity will be the change in the distance Δs divided by the change in time Δt.

$$\frac{\Delta s}{\Delta t}=\frac{s(4)-s(0)}{4-0}=\frac{\left[300-16(4)^2\right]-\left[300-16(0)^2\right]}{4}=\frac{44-300}{4}=\frac{-256}{4}=-64.$$

Thus, the average velocity of the diving hawk is 64 feet per second. Minus sign indicates that the hawk is diving in downward direction.

Example 09: (ROCKET VELOCITY) A toy rocket is shot straight up from the ground and travels so that its distance from the ground after t seconds is $s = 200t - 16t^2$ feet. What is the velocity of the rocket after 2 seconds have passed?

Solution: The velocity v is given by:

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(200t - 16t^2\right) = 200 - 32t.$$

After 2 seconds the velocity of the rocket $v = 200 - 32(2) = 136$ feet per second.

Example 10: (FALLING OBJECT) A brick comes loose from near the top of a building and falls such that its distance s (in feet) from the street (after t seconds) is given by $s=150-16t^2$. How fast is the brick falling after 3 seconds have passed?

Solution: The velocity after t seconds have been passed is:

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(150 - 16t^2\right) = -32t.$$

Now, at t=3, we have $v = ds/dt = -32(3) = -96.$

Thus, the brick is falling with velocity of 96 feet per second (downward) after 3 seconds.

Example 11: (VELOCITY OF A CAR) A racing car begins a short test run and travels according to $s = 8t^2 + t^3/3$, where s is the distance traveled in feet and t is the time in seconds. What is the velocity of the car after 3 seconds have passed?

Solution: The velocity of the car is:

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(8t^2 + \frac{1}{3}t^3\right) = 16t + 3\left(\frac{1}{3}\right)t^2 = 16t + t^2$$

Now, after 3 seconds it is: $v_{t=3} = \left[\frac{ds}{dt}\right]_{t=3} = 16(3) + (3)^2 = 57$ feet/sec.

Example 12: (BACTERIA GROWTH) A colony of 1000 bacteria is introduced to a growth-inhibiting environment and grows according to the formula $n = 1000 + 20t + t^2$, where n is the number of bacteria present at any time t (t is measured in hours). (a) According to the formula, how many bacteria are present at the beginning? (b) What is the rate of growth of the bacteria at any time t? (c) What is the rate of growth after 3 hours? (d) How many bacteria are present after 3 hours?

Solution: (a) The number of bacteria present at the beginning is:

$$n = 1000 + 20(0) + (0)^2 = 1000.$$

(b) The rate of growth of the bacteria at any time t will be:

$$v = \frac{dn}{dt} = \frac{d}{dt}\left(1000 + 20t + t^2\right) = 20 + 2t.$$

(c) The rate of growth after 3 hours will be:

$$v = dn/dt = 20 + 2t = 20 + 2(3) = 26, \text{ bacteria/hour.}$$

(d) The number of bacteria present after 3 hours is:

$$n = 1000 + 20(3) + (3)^2 = 1000 + 60 + 9 = 1069.$$

Example 13: (SALES) Suppose that in June a chain of stores had combined daily sales of ice cream cones given by $s = -0.01x^2 + 0.48x + 50$, where s is the number of hundreds of ice cream cones sold and x is the day of the month. (a) How many ice cream cones were sold by the chain on June 3? (b) At what rate were sales changing

on June 10? (c) At what rate were sales changing on June 28? (d) On what day was the rate of change of sales equal to 10 cones per day?

Solution: (a) On June 3, the number of ice cream cones sold is:

$$s = [-0.01(3)^2 + 0.48(3) + 50] \times 100 = 5135$$

(b) Differentiate s w.r.t x, we get:

$$ds/dx = -0.02x + 0.48. \qquad (1)$$

Put x = 10, and multiplying by 100, we obtain,

$$ds/dx = 0.28\ (100) = 28.$$

This shows that on 10th June the rate of increase in the sale of ice cream cones is 28.

(c) Putting x = 28 in equation (1) and multiplying by 100, we get

$$ds/dx = -0.08\ (100) = -8.$$

This shows that on 28th June, the rate of sale of ice cream cone decreases by 8.

(d) Putting ds/dx equal to 10 in equation (1) and dividing the right hand side by 100, we get:

$$10 = (-0.02x + 0.48)/100 \rightarrow x = 4.76 \sim 5$$

This shows that on 5th June the rate of change of sale of sale will be 10 cones per day.

Example 14: (VOLUME) The volume V of a spherical balloon with radius r is $V = 4\pi r^3/3$. Air is blown into the balloon, both the radius and the volume of the balloon increase. Determine the rate of the change of the volume with respect to the radius when the radius is 10 centimeters.

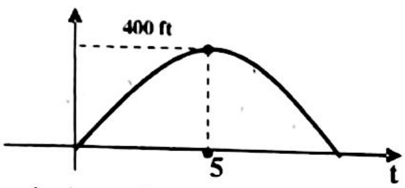
Solution: The rate of the change of the volume with respect to the radius will be;

$$\frac{dV}{dt} = \frac{4\pi}{3}.3r^2 = 4(3.14159)r^2 = 12.56\,r^2$$

When r = 10cm, $\frac{dV}{dt} = 12.56(100) = 1256\ cm^3$.

Example 15: A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109mph). It reaches a height of s = 160t – 16t² feet after t seconds. (a) How high does the rock go? (b) What is the velocity and speed of the rock when it is 256 feet above the ground on the way up and on the way down? (c) What is the acceleration of the rock at any time t during its flight (after the blast)? (d) When does the rock hit the ground again?

Solution: (a) In the coordinate system we have chosen s as height from the ground up, so the velocity is positive on the way up and negative on the way down. The time when the rock is at the highest point during the flight its velocity is zero.



Therefore, to find the maximum height, all we need to do is to find t when v = 0 and evaluate s at this time. Now at any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(160t - 16t^2\right) = 160 - 32t \text{ feet/sec.}$$

Equate this to zero, we get: $160 - 32t = 0 \Rightarrow t = 5\text{sec}$.

This means that rock reaches at maximum height after 5 seconds which is,

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400 \text{ feet.}$$

(b) To find the rock's velocity at 256 feet on the way up and on the way down, we find the two values of t for which $s(t) = 160t - 16t^2 = 256$. Solving, we get

$$160t - 16t^2 = 256 \Rightarrow -16t^2 + 160t - 256 = 0 \Rightarrow t^2 - 10t + 16 = 0 \Rightarrow t = 2 \text{ or } 8.$$

Two values of t show that after the blast, the rock is at the height of 256 feet after 2 seconds on the way up and at the same height after 8 seconds when the rock is on the way back to the ground. The rock's velocities at these times are:

$$v(2)=160-32(2)=96 \text{ ft/sec} \quad \text{and} \quad v(8)=160-32(8)=-96 \text{ ft/sec.}$$

Thus at both instants, the rock's speed is 96 ft/sec.

(c) At any time during its flight following the explosion, the rock's acceleration is,

$$a=\frac{dv}{dt}=\frac{d}{dt}(160-32t)=-32 \text{ ft/sec}^2.$$

Thus, the acceleration of the rock is -32 ft/sec. Minus sign indicates that the direction is against the gravitational force.

(d) When the rock hits the ground, the height is zero, that is; s = 0. So,

$$160t-16t^2=0 \Rightarrow t=0 \text{ or } t=10.$$

Hence, after 10 seconds it hits the ground again.

Example 16: If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R–ohm resistor, the value of R can be found from the equation $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$

If R_1 is decreasing at the rate of 1 ohm/sec, and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when R_1 = 75 ohms and R_2 50 ohms?

Solution: Given

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2} \qquad (1)$$

$$\frac{dR_1}{dt}=-1 \text{ ohm/sec}, \quad \frac{dR_2}{dt}=0.5 \text{ ohm/sec}, \quad R_1=75 \text{ ohms}, \quad R_2=50 \text{ ohms}$$

From (1) we have $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{R_2+R_1}{R_1R_2} \Rightarrow R=\frac{R_1R_2}{R_1+R_2}$ (2)

Differentiating (2) with respect to t, we get

$$\frac{dR}{dt}=\frac{d}{dt}\left(\frac{R_1R_2}{R_1+R_2}\right)=\frac{(R_1+R_2)\frac{d}{dt}(R_1R_2)-R_1R_2\frac{d}{dt}(R_1+R_2)}{(R_1+R_2)^2}$$

$$=\frac{(R_1+R_2)\left(R_1\frac{dR_2}{dt}+R_2\frac{dR_1}{dt}\right)-R_1R_2\left(\frac{dR_1}{dt}+\frac{dR_2}{dt}\right)}{(R_1+R_2)^2}$$

Substituting all the values, we get

$$\frac{dR}{dt}=\frac{(50+75)[75(0.5)+50(-1)]-(75)(50)(-1+0.5)}{(50+75)^2}=0.02 \text{ ohms/sec}.$$

Hence R is changing at the rate of 0.02 ohm/sec.

Related Rates

In physical and social sciences many variables occur which are functions of time. Problems involving rates of change of such variables with respect to time are called "**Related rates problems**". Methods of solutions of such problems are illustrated by following examples.

Example 17: If a tumor is approximately spherical in shape, its volume is approximately $V = 4\pi r^3/3$. The radius of a tumor is growing in an animal is increasing at a rate of 1.25 millimeters per month. Determine how fast the volume of the tumor is increasing when the radius is 10 millimeters?

Solution: Given that $dr/dt = 1.25$ mm/month, $r = 10$mm, we have to find dV/dt.

Now, $$V = 4\pi r^3/3 \qquad (1)$$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4(3.14)(100)(1.25) \Rightarrow \frac{dV}{dt} = 1570\text{mm}^3/\text{month}.$$

Example 18: A cylindrical Can with radius 6 inches and height 20 inches is completely filled with water. Suddenly, it is punctured at the bottom, after which the water pours out at the rate of 12 cubic inches per second. How fast is the water level falling?

Solution: The volume of cylinder is given by given $V = \pi r^2 h$. Here $r = 6$ inches, $h = 20$ inches, $dV/dt = 12$ inch^3/sec.

We have to compute $dh/dt = ?$

Putting the value of $r = 6$, we get, $V = 36\pi h$.

Differentiating with respect to time, we get

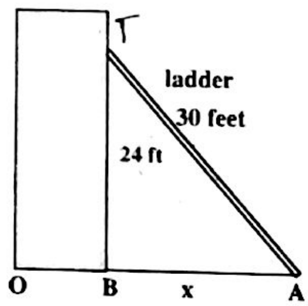
$$\frac{dV}{dt} = 36\pi\frac{dh}{dt} \Rightarrow 12 = 36(3.14)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{113.04} \approx 0.11 \text{ inches/sec.}$$

Thus, the water level is falling (approximately) at the rate of 0.1 inches/sec

Example 19: A 30 feet ladder is leaning against a building. Suppose the ladder is sliding down the wall in such a way that the bottom of the ladder is moving away from the wall at the rate of 2 feet per second. At what rate is the top of the ladder is sliding down the wall when the top of the ladder is 24 feet above the ground?

Solution: Let T be the top of ladder leaning against the wall

Let $|BT| = y$, $|AB| = x$ and $z = |AT|$ feet . Initially we are given that $z = 30$ ft, $y = 24$ ft and $dx/dt = 2$ ft/sec and we have to find dy/dt at the instant when ladder starts sliding down. From the right-angled triangle ABT, we have



$$x^2 + y^2 = z^2 \Rightarrow x^2 + (24)^2 = (30)^2 \Rightarrow x = 18$$

Now reusing equation: $x^2 + y^2 = (30)^2$

Now differentiating equation w.r.t time t, we get:

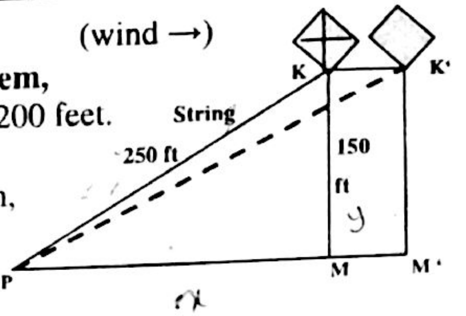
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \frac{d}{dt}(30)^2 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Substituting the values, we get: $(18)(2) + (24)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{36}{24} = -1.5$

Thus, top of the ladder is sliding down at the rate of 1.5 feet/sec. Minus sign shows that the ladder is sliding down so the distance y decreases.

Example 20: A kite is flying 150 feet high, where the wind causes it to move horizontally at the rate of 5 feet per second. In order to maintain the kite at a height of 150 feet, the person must allow more string to be let out. At what rate is the string being let out when the length of the string already out is 250 feet?

Solution: Let P be the position of person flying the kite, K be the position of kite and K' be its position after the string is let out. Other points are shown in the figure.



Now, z = |PK| = 250 and |MK| = 150 feet.

Let |PM| = x, |MK| = y. Thus, by Pythagoras theorem,

$|PK|^2 = |PM|^2 + |MK|^2$ → $(250) = x^2 + (150)^2$ → x = 200 feet.

It may be noted that wind bellows in the x-direction.

Thus dx/dt = 5 ft/sec. Now, using Pythagoras theorem, once again, that is, $z^2 = x^2 + y^2 = x^2 + (150)^2$

Differentiating w.r.t t, we get:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 0 \Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{200}{250}(5) = 4 \text{ ft/sec}.$$

This means that if the string is let out at the speed of 4 ft/sec, the position of kite will be maintained at 150 ft above the ground.

3.5 MARGINAL ANALYSIS

The managers of a manufacturing operation are concerned about the total cost of maintaining a particular level of production. In other words, they want to know the cost C(x) of producing x units. Furthermore, when a particular level of production is being maintained, it is important to know the cost of producing one additional unit. For example, if 100 TV sets are produced, what will it cost to make one more- the 101^{th} TV? Such information assists management in taking marketing decisions about production.

The rate of change interpretation of the derivative leads to a calculus application here. If C(x) is the total cost of producing x units, then C'(x) is the rate of change of the total cost and gives the approximate cost of producing one additional unit. C'(x) is called the **marginal cost**.

Similarly, if P(x) and R(x) are the profit and revenue functions, then P'(x) and R'(x) are known as **marginal profit** and **marginal revenue** respectively.

Example 01: Suppose the cost of producing x units is $C(x) = 100 + 30x - x^2$ dollars (for $0 \le x \le 12$). Determine the marginal cost when x = 9 units.

Solution: The marginal cost is: C'(x) = 30 – 2x. For x = 9, we have

$$C'(9) = 30 - 2(9) = 12$$

Thus marginal cost when x = 9 is \$12. This means that after 9 units have been produced, the cost of producing the next unit (the 10^{th} unit) will be approximately \$12.

It may be noted that **exact cost** of producing the tenth unit can be computed as

$C(10) - C(9) = [100 + 30(10) - (10)^2] - [100 + 30(9) - (9)^2] = 300 - 289 = \$11$.

This is approximately equal to \$12.

Example 02: A furniture manufacturer determines that the marginal cost for making office tables is always increasing. The company decides to stop table production when the marginal cost reaches \$110. Assuming the cost function for table is $C(x) = 0.01x^2 + 80x + 100$ dollars, how many tables will the company make before it halts table production?

Solution: The cost is given by $C(x) = 0.01x^2 + 80x + 100$. Hence, the marginal cost is:

$$C'(x) = 0.02x + 80$$

Given that the marginal cost is \$110 → 110 = 0.02x + 80 → x = 1500.

This means that before further production of table is stopped, the furniture manufacturer must produce 1500 tables in order to get marginal cost equal to 110 dollars.

Example 03: The cost of producing x deep-tread radial tires is $C(x) = 4000 + 70x - 0.01x^2$ dollars, and the revenue from the sale of x tires is $R(x) = 105x - 0.02x^2$ dollars.

(a) Determine marginal cost.

(b) Determine marginal revenue.

(c) Determine MR(50) and tell what it means.

(d) Determine the marginal profit.

(e) For what value of x is the marginal cost equal to the marginal revenue, and what is the marginal profit in that instance?

Solution: (a) Marginal cost MC = C'(x) = 70 – 0.02x dollars.

(b) Marginal revenue MR(x) = R'(x) = 105 – 0.04x dollars.

(c) From (b) it follows that MR(50) = 105 – 0.04(50) = 103 dollars.

This means that once 50 tires have been sold, the revenue to be obtained from the sale of the next tire (the 51st) is approximately $103.

(d) The profit function is given by: $P(x) = R - C = -0.01x^2 + 35x - 4000$

Differentiating yields the marginal profit: MP = P'(x) = -0.02x + 35 dollars.

(e) The marginal cost is 70 – 0.02x and marginal revenue is 105 – 0.04x. If they are equal then: 70 – 0.02x = 105 – 0.04x → x = 1750

The marginal profit is then: MP(1750) = -0.02(1750) + 35 = 0. Thus, the marginal profit is zero when x =1750 tires. You should not be particularly surprised; because P'(x) = R'(x) - C'(x) and R'(x) - C'(x) = 0 when marginal cost and marginal revenue are equal in which case MP is always zero.

Example 04: If the revenue function for a product is $R(x) = 60x^2/(2x + 1)$ find the marginal revenue.

Solution: The marginal revenue is found by differentiating R(x). Now,

$$R'(x)=\frac{d}{dx}R(x)=\frac{d}{dx}\left(\frac{60x^2}{2x+1}\right)=\frac{(2x+1)(120x)-60x^{22}(2)}{(2x+1)^2}=\frac{240x^2+120x-120x^2}{(2x+1)^2}$$

$$=\frac{120x^2-120x}{(2x+1)^2}=\frac{120x(x-1)}{(2x+1)^2}.$$

Example 05: If the total revenue function is given by R(x) = 60x and the total cost function is given by $C(x) = 200 + 10x + 0.1x^2$, what is the marginal profit at x=10?

Solution: The profit function is given as: $P(x) = R - C = -200 + 50x - 0.1x^2$

Now the marginal profit is:

$$P'(x)=\frac{d}{dx}P(x)=\frac{d}{dx}\left(-200+50x-0.1x^2\right)=50-0.2x$$

When x = 10, $P'(10)=50-0.2(10)=48.$

Example 06: If the total revenue function for a commodity is $R = 40x - 0.02x^2$, with x representing the number of units.

(a) Find the marginal revenue function.

(b) At what level of production will marginal revenue be 0?

Solution: (a) differentiating R w.r.t x, we get MR = R`(x) = 40 – 0.04x

(b) The level of production at which the marginal revenue will be zero can be found as:

40 – 0.04x = 0 → x = 1000

Hence, 1000 number of units must be produced in order to have zero marginal revenue. This means that there will be no increase in the revenue if 1000 units are produced, that is, revenue will remain constant.

Example 07: Suppose the cost of producing x units is given by $C(x) = 200 + 15x - 0.5x^2$ dollars, for $0 \le x \le 12$.

(a) Determine the marginal cost when x=7 units.

(b) Determine the exact cost of the 8th unit.

(c) What is meaning of MC(7)?

Solution: (a) The marginal cost is:

MC(x) = C'(x) = 15 – x. At x=7, MC(7) = 15 – 7 = 8

(b) The exact cost of 8th unit is given by:

$C(8) - C(7) = [(200 + 15(8) - 0.5(64)) - (200 + 15(7) - 0.5(49)] = \$7.5$

(c) MC(7) means that once 7 units are produced, the cost of producing the next unit (the 8^{th}) is approximately \$8.

Example 08: Let the revenue function for a stereo system is $R(x) = 300x - x^2$, where x denotes the number of units sold.

(a) What is the marginal revenue if 50 units are sold?

(b) What is the marginal revenue if 100 units are sold?

(c) What is the marginal revenue if 150 units are sold?

(d) What is happening to revenue when 150 units are sold?

Solution: (a) $MR(x) = R`(x) = 300 - 2x$. At x=50, $MR(50) = 300 - 2(50) = 200$

Thus, marginal revenue is \$200/unit if 50 units are sold.

(b) At x =100, $MR(100) = 300 - 2(100) = 100$ dollars/unit

(c) At x =150, $MR(150) = 300 - 2(150) = 0$. This means that if 150 units are sold the marginal revenue will be zero per unit.

(d) $R(150) = 300(150) - (150)^2 = \$22,500$. This means that after the sale of 150 stereos, the revenue will be \$22, 500

3.6 HIGHER DERIVATIVES

So far we have studied the first derivative of a function. Since the derivative of a function is itself a function, so we can find the derivative of the derivative of a function. The derivative of a first derivative is called a second derivative. We can find the second derivative of a function f by differentiating it twice.

Let y = f(x) be a differentiable function on an open interval (a, b). If we apply the definition of derivative to $f'(x)$, the resulting limit (if it exists) will be the second derivative of y = f(x) and is denoted by $y'' = f''(x)$. Thus,

$$y'' = f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Continuing in this way, we can evaluate the third, fourth and higher derivatives of y = f(x), whenever they exist. The successive derivatives of y = f(x) are denoted as follows:

$1^{st}: \; y' = f'(x) = D_x y = \frac{dy}{dx}$ $\qquad$ $2^{nd}: \; y'' = f''(x) = D^2{}_x y = \frac{d^2y}{dx^2}$

$3^{rd}: \; y''' = f'''(x) = D^3{}_x y = \frac{d^3y}{dx^3}$ $\qquad$ $4^{th}: \; y^{(4)} = f^{(4)}(x) = D^4{}_x y = \frac{d^4y}{dx^4}$

....

$(n-1)^{st}: y^{(n-1)} = f^{(n-1)}(x) = D^{n-1}{}_x y = \frac{d^{n-1}y}{dx^{n-1}}$

$n^{th}: y^{(n)} = f^{(n)}(x) = D^n{}_x y = \frac{d^n y}{dx^n}$

Example 01: If $y = \tan^{-1} x$, show that $(1 + x^2) y'' + 2xy' = 0$

Solution: Given that $y = \tan^{-1} x$. Differentiating twice, we get

$$y' = \frac{1}{1+x^2} \quad \Rightarrow (1+x^2)y' = 1$$

Differentiating again, we get:

$$(1+x^2)y'' + y'(0+2x) = 0 \quad \Rightarrow (1+x^2)y'' + 2xy' = 0$$

Example 02: If $y = \ln\left(x+\sqrt{1+x^2}\right)$, prove that $\left(1+x^2\right)y'' + xy' = 0$

Solution: Differentiating both sides of givn equation w.r.t x, we get:

$$y' = \frac{1}{\left(x+\sqrt{1+x^2}\right)}\cdot\frac{d}{dx}\left(x+\sqrt{1+x^2}\right) = \frac{1+\frac{1}{2}\left(1+x^2\right)^{-1/2}.2x}{\left(x+\sqrt{1+x^2}\right)} = \frac{\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$$

➔ $y' = \frac{1}{\sqrt{1+x^2}}$ $\Rightarrow \sqrt{1+x^2}\,y' = 1$. Differentiating once again, we get

$$\sqrt{1+x^2}\,y'' + \frac{1}{2}\left(1+x^2\right)^{-1/2}.2xy' = 0 \quad \Rightarrow \sqrt{1+x^2}\,y'' + \frac{xy'}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \quad \left(1+x^2\right)y'' + xy' = 0$$

Example 03: If $y = \sin(\sin x)$, prove that $y'' + (\tan x)\ y' + y\cos^2 x = 0$

Solution: Given that $y = \sin(\sin x)$, differentiating twice w.r.t. x, we get

$y' = \cos(\sin x).\cos x$

➔ $y'' = \cos(\sin x).(-\sin x) + \cos x\,[-\sin(\sin x).\cos x]$

➔ $y'' = -\sin x.\cos(\sin x) - \cos^2 x\sin(\sin x)$

➔ $y'' = -\cos x\cos(\sin x).\frac{\sin x}{\cos x} - \cos^2 x\sin(\sin x) = -y'\tan x - y\cos^2 x$

Notice that, $\cos x\cos(\sin x) = y'$ and $\sin(\sin x) = y$

➔ $y'' + \tan x\ y' + y\cos^2 x = 0$

Example 04: If $x = \sin t$ & $y = \sin pt$, show that $(1-x^2)\,y'' - x\,y' + p^2 y = 0$

Solution: Given equations are parametric equations, hence

$\frac{dy}{dt} = p\cos pt$ and $\frac{dx}{dt} = \cos t$. Now

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = p\cos pt \div \cos t = \frac{p\cos pt}{\cos t}$$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{p\cos pt}{\cos t}\right)$

$$= \frac{d}{dt}\left(\frac{p\cos pt}{\cos t}\right)\cdot\frac{dt}{dx} = \frac{\cos t\left(-p^2\sin pt\right) - p\cos pt\left(-\sin t\right)}{\cos^2 t} \div \frac{dx}{dt}$$

Multiply by $\cos^2 t$, we get

$$\cos^2 t\,\frac{d^2y}{dx^2} = \left[-p^2\cos t\sin pt + p\cos pt\,\sin t\right] \div \cos t$$

$$\Rightarrow \left(1-\sin^2 t\right)\frac{d^2y}{dx^2} = -\frac{p^2\cos t\sin pt}{\cos t} + \frac{p\cos pt\sin t}{\cos t}$$

$$\Rightarrow \left(1-\sin^2 t\right)\frac{d^2y}{dx^2} = -p^2\sin pt + \left(\frac{p\cos pt}{\cos t}\right)\sin t. \text{ NB} : \sin t = x,\ \sin pt = y,\ \frac{p\cos pt}{\cos t} = \frac{dy}{dx}$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx} \quad \Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

Some Standard n^{th} Order Derivatives

In this section we shall present you general rules of finding the n^{th} order derivatives of some standard functions. These rules are not applicable for every function.

(1) To find the n^{th} derivative of x^m

Let $y = x^m$. Then its successive derivatives are:

$y' = mx^{m-1}, y'' = m(m-1)x^{m-2}, y''' = m(m-1)(m-2)x^{m-3}, \cdots$. Thus,

$y^{(n)} = m(m-1)(m-2)\cdots[m-(n-1)]x^{m-n}$ or $y^{(n)} = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

$$y^{(n)} = \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(m-n-1)\cdots3\cdot2\cdot1}{(m-n)(m-n-1)\cdots3\cdot2\cdot1}x^{m-n}$$

$\Rightarrow y^{(n)} = \dfrac{m!}{(m-n)!}x^{m-n}$, where m is a positive integer and $n < m$.

If $n = m$, then: $y^{(n)} = \dfrac{n!}{(n-n)!}x^{n-n} = \dfrac{n!}{0!}x^0 = n!$, a constant.

If $n > m$, then all successive derivatives will be zero.

Example 05: Let $y = x^6$ then find $y^{(5)}$, $y^{(6)}$ and $y^{(7)}$.

Solution: $y^{(5)} = \dfrac{6!}{(6-5)!}x^{6-5} = 720x$, $\quad y^{(6)} = 6! = 720$, $\quad y^{(7)} = 0$.

(2) To find the n^{th} derivative of $(ax + b)^m$

Let $y = (ax + b)^m$. Then its successive derivatives are

$y' = m(ax+b)^{m-1}a, y'' = m(m-1)(ax+b)^{m-2}a^2, y''' = m(m-1)(m-2)(ax+b)^{m-3}a^3$

Continuing this way, we get

$$y^{(n)} = m(m-1)(m-2)\cdots[m-(n-1)](ax+b)^{m-n}\cdot a^n$$
$$= m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{m-n}\cdot a^n$$
$$= \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(m-n-1)\cdots3\cdot2\cdot1}{(m-n)(m-n-1)\cdots3\cdot2\cdot1}(ax+b)^{m-n}\cdot a^n$$

$\Rightarrow y^{(n)} = \dfrac{m!\,a^n}{(m-n)!}(ax+b)^{m-n}$, $\quad n < m$.

If $n = m$, then $y^{(n)} = \dfrac{n!}{(n-n)!}(ax+b)^{n-n}\cdot a^n = n!a^n$.

If $n > m$ then $y^{(n)} = 0$

Example 06: Let $y = (3x + 4)^7$, then find $y^{(5)}$, $y^{(7)}$, $y^{(8)}$

Solution: Using above formula, we get

$$y^{(5)} = \frac{7!}{(7-5)!}(3x+4)^{7-5}\cdot3^5 = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot3^5}{2\cdot1}(3x+4)^2 = 612360(3x+4)^2.$$

$$y^{(7)} = \frac{7!}{(7-7)!}(3x+4)^{7-7}\cdot3^7 = 7!\times3^7 = (5040)(2187) = 11022480 \quad\Rightarrow y^{(8)} = 0.$$

3. To find the n^{th} derivative of $1/(ax + b)$

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$.

We know that if $y = (ax+b)^m$ then: $y^{(n)} = \frac{m!\,a^n}{(m-n)!}(ax+b)^{m-n}$

Substituting m = -1, we get

$$y^{(n)} = (-1)(-2)(-3)\cdots(-n)(ax+b)^{-1-n}\cdot a^n = (-1)^n\cdot 1\cdot 2\cdot 3\cdots n\,(ax+b)^{-(n+1)}\cdot a^n$$

➔
$$y^{(n)} = \frac{(-1)^n\,n!a^n}{(ax+b)^{n+1}}$$

Example 07: Let y = 1/(5x – 3) then find $y^{(4)}$

Solution: Putting n = 4 in the above equation, we get:

$$y^{(4)} = \frac{(-1)^4\,4!\,5^4}{(5x-3)^{4+1}} = \frac{4\cdot 3\cdot 2\cdot 1\cdot 5^4}{(5x-3)^5} = \frac{15,000}{(5x-3)^5}.$$

4. To find the n^{th} derivative of ln(ax + b)

Let y = ln(ax + b). Differentiating, we get

$y' = a/(ax+b) = a(ax+b)^{-1}$ ➔ $y'' = a^2(-1)(ax+b)^{-2}$

➔ $y''' = a^3(-1)(-2)(ax+b)^{-3}$

➔ $y^{iv} = a^4(-1)(-2)(-3)(ax+b)^{-4} = (-1)^3 3!a^4/(ax+b)^4$

Continuing this way n- times, we get: $y^{(n)} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

Example 08: Let y = ln(2x + 3) then find $y^{(10)}$

Solution: We know that if $y = \ln(ax+b)$ then $y^{(n)} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

Using this formula on y = ln(2x + 3) and putting n = 10, we get:

$$y^{(10)} = \frac{(-1)^9\,(9)!2^{10}}{(2x+3)^{10}} = -\frac{2^{10}.9!}{(2x+3)^{10}}$$

Example 09: Find the n^{th} derivative of:

(a) $y = \frac{1}{a^2 - x^2}$ (b) $y = \frac{1}{x^2 + a^2}$ (c) $y = \tan^{-1}\left(\frac{x}{a}\right)$ (d) $y = \tan^{-1}\frac{2x}{1-x^2}$

Solution: (i) Given that $y = \frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)}$ (1)

Decomposing it into partial fractions, we get

$$\frac{1}{(a+x)(a-x)} = \frac{A}{(a+x)} + \frac{B}{(a-x)} \quad (2)$$

Multiplying by (a + x) (a – x), we get

$$1 = A(a-x) + B(a+x) \quad (3)$$

Let a - x = 0 ➔ x = a. Put this in (3), we get,

$$1 = A(a-a) + B(a+a) \Rightarrow 1 = 2aB \Rightarrow B = 1/2a$$

Let a + x = 0 ➔ x = -a. Put this in (3), we get

$$1 = A(a+a) + B(a-a) \Rightarrow 1 = 2aA \Rightarrow A = 1/2a.$$

Thus, equation (2) becomes:

$$y=\frac{1}{(a+x)(a-x)}=\frac{1}{2a(a+x)}+\frac{1}{2a(a-x)}\Rightarrow y=\frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$$

Differentiating n times using the above formula, we get

$$\frac{d^n y}{dx^n}=\frac{1}{2a}\left[\frac{d^n}{dx^n}\left(\frac{1}{a+x}\right)+\frac{d^n}{dx^n}\left(\frac{1}{a-x}\right)\right]=\frac{1}{2a}\left[\frac{n!(-1)^n}{(a+x)^{n+1}}+\frac{n!(-1)^n}{(a-x)^{n+1}}\right]$$

$$y^{(n)}=\frac{n!}{2a}\left[\frac{(-1)^n}{(a+x)^{n+1}}+\frac{(-1)^n}{(a-x)^{n+1}}\right]=\frac{n!(-1)^n}{2a}\left[\frac{1}{(a+x)^{n+1}}+\frac{1}{(a-x)^{n+1}}\right]$$

(b) Given that $y=\dfrac{1}{x^2+a^2}=\dfrac{1}{(x-ia)(x+ia)}$ (1)

Decomposing it into partial fractions, we get

$$\frac{1}{(x-ia)(x+ia)}=\frac{A}{(x-ia)}+\frac{B}{(x+ia)} \quad (2)$$

Solving as above, we get: A = 1/2i and B = -1/2i

Thus, equation (2) becomes: $y=\dfrac{1}{(x-ia)(x+ia)}=\dfrac{1}{2ia}\left(\dfrac{1}{x-ia}-\dfrac{1}{x+ia}\right)$

Differentiating n times using the above formula, we get

$$\frac{d^n y}{dx^n}=\frac{1}{2ia}\left[\frac{d^n}{dx^n}\left(\frac{1}{x-ia}\right)+\frac{d^n}{dx^n}\left(\frac{1}{x+ia}\right)\right]=\frac{1}{2ia}\left[\frac{n!(-1)^n}{(x-ia)^{n+1}}-\frac{n!(-1)^n}{(x+ia)^{n+1}}\right]$$

$$y^{(n)}=\frac{n!(-1)^n}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]$$

(c) $y=\tan^{-1}\left(\dfrac{x}{a}\right)\Rightarrow y_1=\dfrac{1}{1+\left(\dfrac{x}{a}\right)^2}\cdot\dfrac{1}{a}=\dfrac{1}{a}\cdot\dfrac{1}{\dfrac{(a^2+x^2)}{a^2}}=\dfrac{1}{a}\cdot\dfrac{a^2}{(x^2+a^2)}=\dfrac{a}{(x^2+a^2)}$

Using the result of part (b), we get

$$y^{(n)}=\frac{an!(-1)^n}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]=\frac{n!(-1)^n}{2i}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]$$

(d) $y=\tan^{-1}\dfrac{2x}{1-x^2}$. Putting $x=\tan\theta$ ➔ $\theta=\tan^{-1}x$.

➔ $y=\tan^{-1}\dfrac{2x}{1-x^2}=\tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)=\tan^{-1}(\tan 2\theta)=2\theta=2\tan^{-1}x$

Differentiating w.r.t. x, we get:

$y_1=2.\dfrac{1}{x^2+1}=2.\dfrac{1}{(x-i)(x+i)}=2\left[\dfrac{A}{x-i}+\dfrac{B}{x+i}\right]$. Using Partial Fractions, we get:

A = 1/2i and B = -1/2i. Thus: $y_1=\dfrac{2}{2i}\left[\dfrac{1}{x-i}-\dfrac{1}{x+i}\right]=\dfrac{1}{i}\left[\dfrac{1}{x-i}-\dfrac{1}{x+i}\right]$

Differentiating (n – 1) times and using formula 4, we get:

$$y_n=\frac{1}{i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}\right]=-i\,(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}\right]$$

$$= -i\,(-1)^n(-1)(n-1)!\left[\frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}\right] = (-1)^n\, i\,(n-1)!\left[\frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}\right]$$

5. To find the n^{th} derivative of e^{ax}

Let $y = e^{ax}$. Differentiating it successively, we get

$$y' = ae^{ax},\; y'' = a^2e^{ax},\; y''' = a^3e^{ax},\cdots,\; y^{(n)} = a^ne^{ax}.$$

Example 10: Let $y = e^{2x}$ then find $y^{(7)}$

Solution: $y = e^{2x} \;\Rightarrow\; y^{(7)} = (2)^7 e^{2x} = 128e^{2x}$

6. To find the n^{th} derivative of $\sin(ax + b)$

Let $y = \sin(ax + b)$. Differentiating successively, we get

$$y' = \cos(ax+b)\cdot a = a\sin\left(ax+b+\frac{\pi}{2}\right) \quad \left[\text{since } \cos\theta = \sin\left(\theta+\frac{\pi}{2}\right)\right]$$

$$y'' = a\cdot\cos\left(ax+b+\frac{\pi}{2}\right)\cdot a = a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+2\frac{\pi}{2}\right)$$

$$y''' = a^2\cdot\cos\left(ax+b+2\frac{\pi}{2}\right)\cdot a = a^3\sin\left(ax+b+2\frac{\pi}{2}+\frac{\pi}{2}\right) = a^3\sin\left(ax+b+3\frac{\pi}{2}\right)$$

Continuing this way n times, we get: $y^{(n)} = a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$

Similarly, if $y = \cos(ax + b)$ then: $\dfrac{d^n}{dx^n}\left[\cos(ax+b)\right] = a^n\cos\left(ax+b+n\frac{\pi}{2}\right)$

Example 11: Find the 4^{th} derivative of $\sin(2x + 3)$ and 5^{th} derivative of $\cos(3x - 4)$

Solution: (i) Let $y = \sin(2x + 3)$. Using the formula that if

$y = \sin(ax+b)$, then $y^{(n)} = a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$

$$y^{(4)} = 2^4\sin\left(2x+3+4\frac{\pi}{2}\right) = 16\sin(2x+3+2\pi) = 16\sin(2x+3)$$

NOTE: $\sin(\theta+2\pi) = \sin\theta$. Similarly,

$$y^{(5)} = 3^5\cos\left(3x+4+5\frac{\pi}{2}\right) = 243\cos\left(3x+4+2\pi+\frac{\pi}{2}\right) = 243\cos\left(4x+5+\frac{\pi}{2}\right)$$

➔ $y^{(5)} = -243\sin(3x+4)$ NB: $\cos(\theta+\pi/2) = -\sin\theta$

7. To find the n^{th} derivative of $e^{ax}\sin(bx + c)$

Let $y = e^{ax}\sin(bx + c)$. Differentiating with respect to x, we get

$$y' = e^{ax}\cos(bx+c)\cdot b + \sin(bx+c)\cdot e^{ax}\cdot a$$

$$y' = e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right] \tag{1}$$

Let $a = r\cos\theta$ (i)

$b = r\sin\theta$ (ii)

Squaring and adding (i) and (ii), we get

$$a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta \Rightarrow a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$$

Now dividing (ii) by (i), we get

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \frac{b}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Substituting theses values into (1), we get

$$y' = e^{ax}\left[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)\right]$$

$$= re^{ax}\left[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta\right]$$

➔ $y' = re^{ax}\left[\sin(bx+c+\theta)\right]$ [NOTE: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$]

Observing that y' is obtained from y merely by multiplying right side by $r = \sqrt{a^2+b^2}$ and increasing the angle by $\theta = \tan^{-1}(b/a)$.

Applying the same rule successively, we get

$$y'' = r^2e^{ax}\sin(bx+c+2\theta)$$

$$y''' = r^3e^{ax}\sin(bx+c+3\theta)$$

Continuing this 'n' times, we get: $y^{(n)} = r^ne^{ax}\sin(bx+c+n\theta)$.

Substituting the values of r and θ, we get:

$$y^{(n)} = \left(a^2+b^2\right)^{n/2} e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly, if $y = e^{ax}\cos(bx+c)$ then $y^{(n)} = \left(a^2+b^2\right)^{n/2} e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$.

Example 12: Find the n^{th} derivative of:

(a) $y = e^{ax}\cos^2x\sin x$ (b) $y = x^2/[(x-1)^2(x+2)]$ (c) $y = x\ln[(x-1)/(x+2)]$

Solution: (a) Given that

$$y = e^{ax}\cos^2x\sin x = e^{ax}\frac{(1+\cos 2x)}{2}\sin x = \frac{e^{ax}}{2}[\sin x + \sin x\cos 2x]$$

$$= \frac{e^{ax}}{2}\left[\sin x + \frac{1}{2}(2\sin x\cos 2x)\right] = \frac{e^{ax}}{2}\left[\sin x + \frac{1}{2}\{\sin(x+2x) + \sin(x-2x)\}\right]$$

$$= \frac{e^{ax}}{2}\left[\sin x + \frac{1}{2}\sin 3x - \frac{1}{2}\sin x\right] = \frac{e^{ax}}{2}\left[\frac{1}{2}\sin x + \frac{1}{2}\sin 3x\right] = \frac{1}{4}\left[e^{ax}\sin x + e^{ax}\sin 3x\right]$$

NOTE: $2\sin a\cos b = \sin(a+b) + \sin(a-b)$

Now we know that if:

$y = e^{ax}\sin bx$ then $y^{(n)} = \left(a^2+b^2\right)^{n/2} e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Using this formula with b = 1 and b = 3, we get:

$$y^{(n)} = \frac{1}{4}\left(a^2+1^2\right)^{n/2} e^{ax}\sin\left(x+n\tan^{-1}\frac{1}{a}\right) + \frac{1}{4}\left(a^2+3^2\right)^{n/2} e^{ax}\sin\left(3x+n\tan^{-1}\frac{3}{a}\right)$$

$$\Rightarrow y^{(n)} = \frac{e^{ax}}{4}\left[\left(a^2+1\right)^{n/2}\sin\left(x+n\tan^{-1}\frac{1}{a}\right) + \left(a^2+9\right)^{n/2}\sin\left(3x+n\tan^{-1}\frac{3}{a}\right)\right]$$

(b) Given that $y = \dfrac{x^2}{(x-1)^2(x+2)} = \dfrac{A}{(x-1)^2} + \dfrac{B}{(x-1)} + \dfrac{C}{(x+2)}$

$= \dfrac{1}{3(x-1)^2} + \dfrac{5}{9(x-1)} + \dfrac{4}{9(x+2)}$. This is by Partial Fraction Case – II

Thus, $y = \frac{1}{3}(x-1)^{-2} + \frac{5}{9}(x-1)^{-1} + \frac{4}{9}(x+2)^{-1}$

Now differentiating successively, we get:

$$y_1 = \frac{1}{3}(-2)(x-1)^{-3} + \frac{5}{9}(-1)(x-1)^{-2} + \frac{4}{9}(-1)(x+2)^{-2}$$

$$y_2 = \frac{1}{3}(-2)(-3)(x-1)^{-4} + \frac{5}{9}(-1)(-2)(x-1)^{-3} + \frac{4}{9}(-1)(-2)(x+2)^{-3}$$

$$y_3 = \frac{1}{3}(-2)(-3)(-4)(x-1)^{-5} + \frac{5}{9}(-1)(-2)(-3)(x-1)^{-4} + \frac{4}{9}(-1)(-2)(-3)(x+2)^{-4}$$

$$= (-1)^3\left[\frac{4!}{3(x-1)^5} + \frac{5.3!}{9(x-1)^4} + \frac{4.3!}{9(x+2)^4}\right]$$

Continuing this n times, we get:

$$y^{(n)} = (-1)^n\left[\frac{(n+1)!}{3(x-1)^{n+2}} + \frac{5.n!}{9(x-1)^{n+1}} + \frac{4.n!}{9(x+2)^{n+1}}\right]$$

(c) Given that $y = x\ln\frac{(x-1)}{(x+1)} = x[\ln(x-1) - \ln(x+1)] = x\ln(x-1) - x\ln(x+1)$

Differentiate w.r.t x, we get:

$$y_1 = x.\frac{1}{x-1} + \ln(x-1) - x.\frac{1}{x+1} - \ln(x+1) = \frac{x-1+1}{x-1} + \ln(x-1) - \frac{x+1-1}{x+1} - \ln(x+1)$$

$$= \frac{x-1}{x-1} + \frac{1}{x-1} - \frac{x+1}{x+1} + \frac{1}{x+1} + \ln(x-1) - \ln(x+1)$$

$$= 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1} + \ln(x-1) - \ln(x+1)$$

$$= \frac{1}{x-1} + \frac{1}{x+1} + \ln(x-1) - \ln(x+1) = (x-1)^{-1} + (x+1)^{-1} + \ln(x-1) - \ln(x+1)$$

Differentiating successively n times w.r.t, we get:

$$y_n = (-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right] + (-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} + \frac{1}{(x+1)^n}\right]$$

REMARK: Here we have directly used formulae (3) and (4).

3.7 LEIBNIZ'S THEOREM

Leibniz's theorem helps us in finding the n^{th} derivative of the product of two functions of x.

Statement: If y = u v, where u and v are functions of x, having derivatives of n^{th} order, then

$$y_n = {}^nC_0u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + {}^nC_3u_{n-3}v_3 + \ldots + {}^nC_ru_{n-r}v_r + \ldots + {}^nC_{n-1}u_1v_{n-1} + {}^nC_nuv_n$$

where suffices of u and v denote the number of times they are differentiated and ${}^nC_r$ denotes the number of combinations of n with r and is given by: ${}^nC_r = \frac{n!}{r!(n-r)!}$.

Using this formula we have following results:

$${}^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1, \qquad {}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n,$$

$${}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2},$$

$${}^nC_3 = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6},\ {}^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1.$$

Proof: By "Mathematical Induction", put n = 1, we get

$$y_1 = u_1 v + u v_1$$

Put n = 2, we have $y_2 = u_2 v + 2 u_1 v_1 + u v_2$

Thus theorem is true for n = 1, 2. Suppose now that the theorem is true for n = k. Then

$$y_k = {}^kC_0 u_k v + {}^kC_1 u_{k-1} v_1 + {}^kC_2 u_{k-2} v_2 + {}^kC_3 u_{k-3} v_3 + \cdots + {}^kC_{k-1} u_1 v_{k-1} + {}^kC_k u v_k$$

Differentiating both sides of the above equation, we obtain

$$y_{k+1} = {}^kC_0(u_{k+1}v + u_k v_1) + {}^kC_1(u_k v_1 + u_{k-1}v_2) + \ldots {}^kC_{k-1}(u_2 v_{k-1} + u_1 v_k) + {}^kC_k(u_1 v_k + u v_{k+1})$$

$$= {}^kC_0 u_{k+1} v + [{}^kC_0 + {}^kC_1]u_k v_1 + [{}^kC_1 + {}^kC_2]u_{k-1} v_2 + \ldots + [{}^kC_{k-1} + {}^kC_k]u_1 v_k + {}^kC_k u v_{k+1}$$

But we know that for all $n \in N$, ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$. Using this result, we get:

$$y_{k+1} = {}^{k+1}C_0 u_{k+1} v + {}^{k+1}C_1 u_k v_1 + {}^{k+1}C_2 u_{k-1} v_2 + \cdots + {}^{k+1}C_k u_1 v_k + {}^{k+1}C_{k+1} u v_{k+1}$$

Replacing k + 1 by n, we get

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_{n-1} u_1 v_{n-1} + {}^nC_n u v_n$$

Hence the result is true for all positive integer n and theorem is proven..

NOTE: It may be noted that the term ${}^nC_r u_{n-r} v_r$ is a general term of the binomial expansion of $(u + v)^n$.

Example 01: Find the n^{th} derivative of the function $y = x^3 e^{ax}$ using Leibniz's Theorem.

Solution: Let $u = e^{ax}, u_n = a^n e^{ax}, u_{n-1} = a^{n-1}e^{ax}, u_{n-2} = a^{n-2}e^{ax}, u_{n-3} = a^{n-3}e^{ax}, \cdots$

Again let $v = x^3, v_1 = 3x^2, v_2 = 6x, v_3 = 6, v_4 = 0$

Using Leibniz's theorem, we have

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3$$

Substituting all the values, an taking e^{ax} common through, we get

$$y_n = e^{ax}\left[a^n x^3 + 3na^{n-1}x^2 + \frac{6n(n-1)a^{n-2}x}{2!} + \frac{6n(n-1)(n-2)a^{n-3}}{3!}\right]$$

$$y_n = e^{ax}\left[a^n x^3 + 3na^{n-1}x^2 + 3n(n-1)a^{n-2}x + n(n-1)(n-2)a^{n-3}\right]$$

Example 02: If y = a cos(ln x) + b sin(ln x), prove that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Proof: We have $y = a\cos(\ln x) + b\sin(\ln x)$. Differentiating, we get

$y_1 = -a\sin(\ln x)\left(\frac{1}{x}\right) + b\cos(\ln x)\left(\frac{1}{x}\right)$. Multiplying both sides by x, we get:

→ $xy_1 = -a\sin(\ln x) + b\cos(\ln x)$. Again differentiating, we get

$$xy_2 + y_1(1) = -a\cos(\ln x)\left(\frac{1}{x}\right) - b\sin(\ln x)\left(\frac{1}{x}\right) = -\frac{1}{x}\left[a\cos(\ln x) + b\sin(\ln x)\right]$$

→ $x^2y_2 + xy_1 = -y$ → $x^2y_2 + xy_1 + y = 0$ (1)

Differentiating (1) n times using Leibniz's theorem, we get

$$\left[{}^nC_0 y_{n+2} x^2 + {}^nC_1 y_{n+1}(2x) + {}^nC_0 y_{n+2}(2)\right] + \left[{}^nC_0 y_{n+1} x + {}^nC_1 y_n (1)\right] + y_n = 0$$

$$x^2 y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$$

$$x^2y_{n+2}+(2n+1)xy_{n+1}+n^2y_n-ny_n+ny_n+y_n=0$$
$$x^2y_{n+2}+(2n+1)xy_{n+1}+\left(n^2+1\right)y_n=0$$

Example 03: Use Leibniz's theorem to show that if $y=\left(x+\sqrt{1+x^2}\right)^k$, then

$$\left(1+x^2\right)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-k^2\right)y_n=0$$

Solution: We have $y=\left(x+\sqrt{1+x^2}\right)^k$, then differentiating it, we get

$$y_1=k\left(x+\sqrt{1+x^2}\right)^{k-1}\left[1+\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]=k\left(x+\sqrt{1+x^2}\right)^{k-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$
$$=k\left(x+\sqrt{1+x^2}\right)^{k-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\Rightarrow\sqrt{1+x^2}\,y_1=k\left(x+\sqrt{1+x^2}\right)^k$$

$\sqrt{1+x^2}\,y_1=ky\Rightarrow\left(1+x^2\right)(y_1)^2=k^2y^2$. Differentiating again, we get

$$\left(1+x^2\right)(2y_1)(y_2)+(y_1)^2(2x)=k^2(2y)(y_1)\Rightarrow\left(1+x^2\right)y_2+xy_1=k^2y \qquad (1)$$

Differentiating (1) n times using Leibniz's theorem, we get

$$\left[{}^nC_0y_{n+2}\left(1+x^2\right)+{}^nC_1y_{n+1}(2x)+{}^nC_2y_n(2)\right]+\left[{}^nC_0y_{n+1}(x)+{}^nC_1y_n(1)\right]=k^2y_n$$
$$\left(1+x^2\right)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-k^2y_n=0$$
$$\left(1+x^2\right)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n-ny_n+ny_n-k^2y_n=0$$
$$\left(1+x^2\right)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-k^2\right)y_n=0.$$

Example 04: If $x=\tan(\ln y)$, prove that $(1+x^2)\,y_{n+1}+(2nx-1)\,y_n+n(n-1)\,y_{n-1}=0$

Solution: Given that $x=\tan(\ln y)$ ➔ $\tan^{-1}x=\ln y$ ➔ $y=e^{\tan^{-1}x}$. Thus,

$$y_1=e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}\Rightarrow\left(1+x^2\right)y_1=e^{\tan^{-1}x}=y \text{ ➔ } \left(1+x^2\right)y_1-y=0.$$

Differentiating n times using Leibnitz's Theorem, we get

$$\left(1+x^2\right)y_{n+1}+n(2x)y_n+\frac{n(n-1)}{2!}.2y_{n-1}-y_n=0.\text{ Simplifying, we get:}$$
$$\left(1+x^2\right)y_{n+1}+n(2x)y_n+\frac{n(n-1)}{2!}.2y_{n-1}-y_n=0$$
$$(1+x^2)\,y_{n+1}+(2nx-1)\,y_n+n(n-1)\,y_{n-1}=0$$

Example 05: If $y=(\sin^{-1}x)^2$, prove that
(i) $y_n(0)=0$ if n is odd and (ii) $y_n(0)=2.\,2^2.\,4^2\ldots(n-2)^2$ if n is even.

Solution: Given that $y=(\sin^{-1}x)^2$, then

$$y_1=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\,y_1=2\sin^{-1}x. \qquad (1)$$

Squaring both sides, we get

$$\left(1-x^2\right)(y_1)^2=4\left(\sin^{-1}x\right)^2=4y$$

Differentiating again w.r.t x, we get

$(1-x^2)\,2y_1.\,y_2+(y_1)^2(-2x)=4\,y_1$. Dividing by $2y_1$, we obtain:

$(1-x^2)\, y_2 - xy_1 - 2 = 0$ (2)

Differentiating this equation n times using Leibnitz's Theorem , we get:

$$\left(1-x^2\right)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - \left[xy_{n+1} + n.1y_n\right] - 0 = 0.$$

Simplifying, we get: $(1 - x^2)\, y_{n+2} - (2nx - 1)x\, y_{n+1} - n^2\, y_n = 0$ (3)

Put x = 0 in (1), we get: $y_1(0) = \sin^{-1}(0) = 0$

Put x = 0 in (2), we get: $y_2(0) = 2$

Put x = 0 in (3), we get: $y_{n+2}(0) = n^2\, y_n(0)$ (4)

Now in (4): Put n = 1, $y_3(0) = 1^2.y_1(0) = 0$
Put n = 3, $y_5(0) = 3^2.y_3(0) = 0$
Put n = 5, $y_7(0) = 5^2.y_5(0) = 0$

This shows that if n is odd, then $y_n(0) = 0$.

Now in (4): Put n = 2, $y_4(0) = 2^2.y_2(0) = 2.2^2$
Put n = 4, $y_6(0) = 4^2.y_4(0) = 2.2^2.4^2$
Put n = 6, $y_8(0) = 6^2.y_6(0) = 2.2^2.4^2.\, 6^2$

This shows that if n is even, then $y_n(0) = 2.2^2.4^2.\, 6^2 \ldots (n-2)^2$

WORKSHEET 03

1. Differentiate the following functions:

$(i)\ f(x) = \frac{1}{2}x^2 + \frac{1}{7}x + \frac{1}{4}$, $(ii)\ f(x) = \left(x + \frac{1}{x}\right)^2$, $(iii)\ f(x) = \left(x^2+3\right)\left(2x^2-1\right)$

$(iv)\ f(x) = \sqrt{x}\,(2x-1)\left(x^2+x+1\right)$, $(v)\ f(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

2. If $f(x) = 6x^2 - 5x + 3$, find $f'(0)$. For what value of x is $f'(x) = 0$

3. Fnd y` where:

(a) $y = x^{\sin y}$ (b) $y = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$ (c) $y = \tan^{-1}\left(\frac{1+2x}{2-x}\right)$

(d) $y = \ln\left(\sin^{-1} e^x\right)$ (f) $y = \left(\sin^{-1} x^2\right)^4$ (g) $y = \left(\cos^{-1} x^2\right)^\pi$

(h) $y = \frac{1-\cos x}{1+\cos x}$ (i) $y = \frac{1-\cosh x}{1+\cosh x}$ (j) $y = \log_{10}\left(\frac{x+1}{x}\right)$

(k) $y = \cos^{-1}\sqrt{1-x^2}$ (l) $y = \sec^{-1}(\sinh x)$ (m) $y = \cosh^{-1}\left(1+x^2\right)$

(n) $\sqrt{x} + \sqrt{y} - \sqrt{2} = 0$ (o) $xy^2 - 2xy + x = 1$ (p) $x^3 + y^3 - 3axy = 0$

(q) $y = \left(x^2+y^2\right)^3$ (r) $\tan^{-1}\left(\frac{y}{x}\right) + yx^2 = 1$ (s) $y = \tan^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)^{1/2}$

(t) $\tan^{-1}(x+y) = \sin^{-1}\left(e^y + x\right)$ (u) $y = \sin^{-1}(\ln x) - \ln\left(\tan^{-1} x\right)$

(v) $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$ [Hint : Put $x = \cos\theta$]

(w) $x = a(t - \sin t)$ and $y = a(1 - \cos t)$, $(x) = 3\cos t - \cos 3t$ and $y = 3\sin t - \sin 3t$

(y) $x = a(t - \sin t)$ & $y = a(1 - \cos t)$ (za) $x = a[\cos t + \ln \tan (t/2)]$ & $y = a \sin t$

(zb) $x = \tan^{-1} \dfrac{2\theta}{1-\theta^2}$ & $y = \sin^{-1} \dfrac{2\theta}{1+\theta^2}$

4. Differentiate logarithmically:

(a) $y = \dfrac{\sqrt{x}\,(1-2x)^{2/3}}{(2-2x)^{3/4}(3-4x)^{4/3}}$ (b) $y = x^x e^x \sin(\ln x)$ (c) $y = e^{\sec^{-1}\left(\frac{1}{x}\right)}$

(d) $y = \dfrac{(x+2)^2}{(x+1)(x^2+3)^2}$ (e) x^x (f) $y = (\sin x)^x$ (g) $y = x^{\sin 2x}$

(h) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$ (i) $y = (\sin x)^{\cos x}$ (k) $y = x^{\sin x} + (\sin x)^x$

5. If $\sin y = x \sin(a + x)$ prove that $y' = \sin^2(a + y)/\sin a$

6. If $x^2 + xy + 3y^2 = 1$, prove that $(x + 6y)^3 y'' + 22 = 0$

7. Differentiate with respect to x each of the following:

(i) $f(x) = e^{ax} \cos(b \arctan x)$ (ii) $f(x) = \ln(\arcsin e^x)$ (iii) $f(x) = \ln(\tanh 2x)$

(iv) $f(x) = \arccos\left(\sqrt{1-x^2}\right)$ (v) $f(x) = \cosh^{-1}(1+x^2)$.

8. Find $\dfrac{dy}{dx}$ in each of the following:

(i) $x^3 + y^3 - 3axy = 0$ (ii) $y = \arcsin(\ln x) - \ln(\arctan x)$

(iii) $\arctan(x+y) = \arcsin(e^y + x)$ (iv) $y \arcsin x - x \arctan y = 1$

9. Find dy/dx in each of the following:

(i) $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ (ii) $y = (x)^{\frac{1}{x}}$ (iii) $y = x^{\ln x}$ (iv) $y = x^{\sin y}$.

10. If $y = e^{ax} \sin bx$, prove that $y'' - 2a y' + (a^2 + b^2) y = 0$

11. The driver of an experimental racing car begins a test run. During the first 6 seconds, the distance s (in feet) of the car from the starting point is

$$s = 14t^2 - t^3/3 \qquad 0 \le t \le 6$$

where t is the number of seconds the car has been moving. What is the velocity of the car after 4 seconds have passed?

12. At time t, the position of a body moving along the s – axis is $s = t^3 - 6t^2 + 9t$ m. (a) Find the body's acceleration each time the velocity is zero. (b) Find the body's speed each time the acceleration is zero. (c) Find the total distance traveled by the body from t = 0 to t = 2.

13. A bicycle manufacturer estimates that it can price its bicycles at $p = 140 - 0.02x^2$ dollars each, where x is the number sold. The cost of producing x bicycles is $900 - 0.01x^2$ dollars. Determine the marginal profit when 20 bicycles are made.

14. The pollution in a lake is being reduced over a 5–year period. The amount of pollutants (in pounds) is given by: $A = 110 - \dfrac{90t}{t+1}$, $0 \le t \le 5$

where t is the time in years.

(a) Determine the amount of pollution in the lake at the beginning, after 2 years, and after 5 years.

(b) Determine the rate at which the pollution in the lake is changing at $t = 2$ years.

(c) What is the meaning of the given sign obtained in part (b)?

15. On Earth, in the absence of air, a rock at a velocity of 24m/sec (about 86km/h) would reach a height of $s = 24t - 4.9t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration at time t.

(b) How long would it take the rock to reach its highest point?

(c) How high would the rock go?

(d) How long would it take the rock to reach half its maximum height?

(e) How long would the rock be aloft?

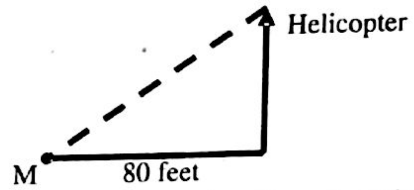
16. A 45–caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high would the bullet go?

17. Suppose an oil spill has taken the form of a circular region and its area is increasing at the rate of 100 square meters per hour. At what rate is the radius of the region increasing when the radius is 200 meters?

18. Consider a spherical balloon with volume $V = 4\pi r^3/3$ that is being inflated by helium at the rate of 4 cubic feet per minute. At what rate is the radius increasing when the radius is 2 feet?

19. A rocket is launched straight up. There is an observation station 7 miles from the launch site. At what rate is the distance between the rocket and the station increasing when the rocket is 24 miles high and traveling at 200 miles per hour?

20. A helicopter rises vertically at the rate of 10 feet per second. There is a maker (M in the figure) 80 feet from where the helicopter lifts off. At what rate is the distance between the helicopter and the marker changing when the helicopter is 192 feet high?



21. A ball is thrown straight up from the ground and travels such that its distance from the ground at any time t is $s = -16t^2 + 80t$ feet. Find its acceleration at any time

22. Find the n^{th} derivative of following:

(a) $y = \dfrac{x}{2x^2+3x+1}$ (b) $y = \dfrac{x}{x^2+a^2}2$ (c) $y = \cos^2 x$ (d) $y = \sin 2x \cos 3x$

(e) $y = e^x \cos^2 x$ (f) $y = \ln x^2$ (g) $y = e^x \sin 4x \cos 6x$

(h) $y = x^3 \ln x$ (i) $y = x^{n-1} \ln x$ (j) $y = e^x \ln x$

(k) $y = e^{4x} \sin(2x+3)$ (l) $y = \ln(2x+3)$

23. If $y = \sin(a \arcsin x)$, prove that $(1-x^2)y_{n+2} = (2n+1)x\, y_{n+1} + (n^2 - a^2)\, y_n$

24. If $y = e^{m \arcsin x}$, show that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2+m^2)y^{(n)} = 0$.

25. Show that: $\dfrac{d^n}{dx^n}\left(\dfrac{\ln x}{x}\right) = \dfrac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \cdots - \dfrac{1}{n}\right]$.

26. Use Leibniz's theorem to find the n^{th} derivative of $y = x^4 e^{2x}$

27. If $y = \sin^{-1}x$ show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. Hence prove that $(1-x^2)y_5 - 7xy_4 - 9y_3 = 0$

28. If $y = \cos(m \ln x)$, show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

29. If $y = \sin^{-1} x / \sqrt{1-x^2}$. then show that $y_{n+2} = (n+1)^2 y_n$

30. If $y = \sinh(m \sinh^{-1}x)$, prove that $(1+x^2)y_2 + xy_1 - m^2 y = 0$

31. If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Hint: $y^{1/m} + y^{-1/m} = 2x \Rightarrow y^{1/m} + \dfrac{1}{y^{1/m}} = 2x \Rightarrow y^{2/m} + 1 = 2xy^{1/m}$

$\Rightarrow y^{2/m} - 2xy^{1/m} + x^2 = x^2 - 1 \Rightarrow (y^{1/m} - x)^2 = x^2 - 1 \Rightarrow (y^{1/m} - x) = \sqrt{x^2-1}$

$y^{1/m} = x + \sqrt{x^2-1} \quad \Rightarrow y = \left(x + \sqrt{x^2-1}\right)^m$.

This is same equation as Example 3 of Leibnitz's Theorem Section.

CHAPTER FOUR

PARTIAL DIFFERENTIATION

4.1 INTRODUCTION

In the previous chapter we dealt with the calculus of functions of one variable and saw what powerful tools of differentiation could be used in assisting with analysis and design in engineering, natural, social sciences and marginal analysis problems? In many engineering and other applications, however, the system or phenomenon we wish to model mathematically depends on more than one variable. For example, the area 'A' of a rectangular metallic plate of width x and breadth y is given by $A = x\,y$.

Since the variables x and y are independent of one another, we say that the dependent variable A (area) is a function of two independent variables x (width) and y (breadth). We express this by writing $A = f(x, y)$ or simply $A(x, y)$. As another example, the life 'L' of an aircraft wing may be modeled by:

$$L = k\,A\,v^2\,\rho$$

where k is a constant, A is the area of the wing, v is the aircraft speed and ρ is the air density. This is an example of a function of three independent variables: A, v and ρ. In this case we write $L = f(A, v, \rho)$ or simply $L(A, v, \rho)$.

More generally, a variable z may be a function of n independent variables $x_1, x_2, x_3,.. x_n$, which we express as: $z = f(x_1, x_2, x_3,.. x_n)$ (1)

The function of n-variables given in (1) has a **domain** in n-dimensional space, a **range** and a **rule** that assigns each n-tuple of real numbers $(x_1, x_2, x_3,.. x_n)$ in the n-dimensional domain with a real number z in the range. Again the rule is frequently expressed in terms of mathematical formulae. Furthermore, a domain may be a restricted region in n-dimensional space; for example if $\tau = \tau(x, y, z)$ represents the temperature of a heated rigid body at a point in the body having coordinates (x, y, z) then the functional relationship has no meaning at points outside the body, so the domain is the set of all points (x, y, z) within the body.

In this section we are primarily concerned with extending the concept of differentiation to functions of more than one variable. As results can be adequately illustrated using functions of two or three independent variables, we shall restrict our attention to these. We now discuss two important concepts namely 'Limit' and 'Continuity' in two independent variables which can be extended to three or more variables.

Limit and Continuity

A function f(x, y) is said to tend to the limit L as $x \to a$ and $y \to b$ if and only if L is independent of the path as $x \to a$ and $y \to b$. In this case we write:

$\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = L$, L being a finite number.

A function f(x, y) is said to be continuous at the point (a, b) if: $\lim\limits_{x\to a\; y\to b} f(x,y)$ exists (finite) and $\lim\limits_{x\to a\; y\to b} f(x,y) = f(a,b)$

Generally, $\lim\limits_{x\to a}\left[\lim\limits_{y\to b} f(x,y)\right] = \lim\limits_{y\to b}\left[\lim\limits_{x\to a} f(x,y)\right]$. But it is not always true.

REMARK: Sometimes it is asked to find the continuity of a given function along some specific path that may be a straight line or parabola or any other curve.

Example 01: Evaluate the following limits.

(i) $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5}$ **(ii)** $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{2xy}{3x^2+y^2}$ **(iii)** $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy}{x^2+2y^2}$

Solution: (i) $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5} = \lim\limits_{x\to 1}\left[\lim\limits_{y\to 2} \dfrac{3x^2y}{x^2+y^2+5}\right] = \lim\limits_{x\to 1}\left[\dfrac{6x^2}{x^2+9}\right] = \dfrac{6}{10} = \dfrac{3}{5}$

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{2xy}{3x^2+y^2} = \lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \dfrac{2xy}{3x^2+y^2}\right] = \lim\limits_{x\to 0}\left[\dfrac{0}{3x^2}\right] = \dfrac{0}{0}$. Thus limit does not exist.

(iii) $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy}{x^2+2y^2} = \lim\limits_{x\to \infty}\left[\lim\limits_{y\to 2} \dfrac{2xy}{3x^2+y^2}\right] = \lim\limits_{x\to \infty}\left[\dfrac{4x}{3x^2+4}\right] = \lim\limits_{x\to \infty}\left[\dfrac{4x}{x^2\left(3+4/x^2\right)}\right]$

$$= \lim_{x\to \infty}\left[\frac{4}{x\left(3+4/x^2\right)}\right] = \frac{4}{\infty} = 0$$

Example 02: If f(x, y) = (x + y)/(2x – y), show that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$

Solution: Consider $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \dfrac{x+y}{2x-y}\right] = \lim\limits_{x\to 0}\left[\dfrac{x}{2x}\right] = \lim\limits_{x\to 0}\left[\dfrac{1}{2}\right] = \dfrac{1}{2}$

Now consider $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} \dfrac{x+y}{2x-y}\right] = \lim\limits_{y\to 0}\left[\dfrac{y}{-y}\right] = \lim\limits_{x\to 0}[-1] = -1$

We observe that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$

Example 03: Show that the function f(x, y) = (x + y)/(x + 2y) is not continuous at (0, 0) along the straight line y = mx.

Solution: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x+y}{x+2y} = \lim\limits_{x\to 0}\left[\dfrac{x+mx}{x+2mx}\right] = \lim\limits_{x\to 0}\left[\dfrac{x(1+m)}{x(1+2m)}\right] = \lim\limits_{x\to 0}\left[\dfrac{(1+m)}{(1+2m)}\right] = \dfrac{1+m}{1+2m}$

Since different values of m produces different values of given limit hence, given function is not continuous at the origin (0, 0).

Alternatively, consider:

$\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \dfrac{x+y}{x+2y}\right] = \lim\limits_{x\to 0}\left[\dfrac{x}{x}\right] = \lim\limits_{x\to 0}(1) = 1$

Also, $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right] = \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} \dfrac{x+y}{x+2y}\right] = \lim\limits_{y\to 0}\left[\dfrac{y}{2y}\right] = \lim\limits_{x\to 0}\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

Since $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$ hence, given function is not continuous at (0, 0).

Example 04: Examine the continuity of the function f(x, y) given below at (0, 0).

$$f(x,y) = \begin{cases} xy/(x^2+y^2), & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Solution: $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\dfrac{xy}{x^2+y^2}\right] = \lim\limits_{x\to 0}\left[\dfrac{0}{x^2}\right] = \dfrac{0}{0}$ (Undefined)

Similarly, $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right] = \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\dfrac{xy}{x^2+y^2}\right] = \lim\limits_{y\to 0}\left[\dfrac{0}{y^2}\right] = \dfrac{0}{0}$ (Undefined)

Since both limits are undefined hence, given function is not continuous at the origin (0, 0).

4.2 PARTIAL DERIVATIVES

Consider a function of two variables $z = f(x, y)$ with $D_f \subset R^2$. If x is changed to $x + \Delta x$ and y remains constant, then the change Δz in z is given by:

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

Dividing each side by Δx and taking the limit Δx tends to zero, we have

$$\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x,y)}{\Delta x}$$

If the limit on right side exists, it is called the **partial derivative** of z w.r.t x and is usually denoted by: $\partial z/\partial x$ or f_x or $\partial f/\partial x$.

The symbol "∂" is known as "dawa". Hence $\partial z/\partial x$ is read as "dawa z over dawa x". It is also read as "partial z over partial x".

Similarly, the partial derivative of $z = f(x, y)$ w.r.t y is

$$\frac{\partial z}{\partial y} = \lim_{\Delta y\to 0}\frac{\Delta z}{\Delta y} = \lim_{\Delta y\to 0}\frac{f(x, y+\Delta y)-f(x,y)}{\Delta y}$$

provided the limit on the right side exists.

The calculation of partial derivatives of a given function $z = f(x, y)$ is quit simple. To obtain f_x, we find the derivative of f w.r.t x holding y constant.

Thus, if: $z = x^2 + y^2 \Rightarrow \dfrac{\partial z}{\partial x} = 2x + 0 = 2x$ and $\dfrac{\partial z}{\partial y} = 0 + 2y = 2y$

and if $z = x^2 y^2 \Rightarrow \dfrac{\partial z}{\partial x} = 2xy^2$ and $\dfrac{\partial z}{\partial y} = 2x^2 y$

Surfaces in Space

A function $z = f(x, y)$ of two variables may be viewed geometrically in one of two ways. One is to draw the level curves, which are curves in the two-dimensional (x, y) plane on which the function takes constant values; that is, the level curves are determined by

$$f(x, y) = c$$

For example, the level curves for the function $z = x^2 + y^2 = c^2$ are concentric circles of radius c, where c is the value of z on the level curve (see fig. 1). Alternatively, in the particular case of a function of two variables, the function (1) may be viewed as a surface in three dimensional space, the surface being obtained by plotting the points corresponding to (x, y, z), with $z = f(x, y)$, using the rectangular Cartesian axes. Such a surface may be built up from the level curves as illustrated in figure 2 for the function $z = x^2 + y^2$.

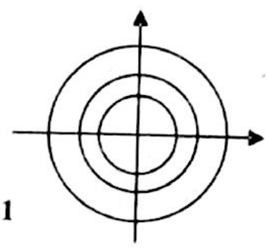


Fig. 1

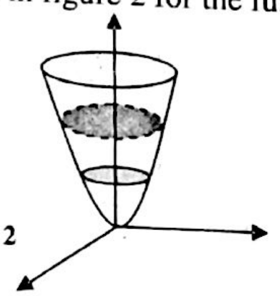


Fig. 2

In the particular example $z = x^2 + y^2$ it is relatively easy to draw its surface. In general, however, plotting such surfaces is not easy. There is now widely available computer software that helps to plot the graphs in 3- dimensions.

Analogously to level curves for functions of two variables, we have level surfaces for functions of three variables: $w = f(x, y, z)$

These are the surfaces on which the function takes constant values, and so are determined by $f(x, y, z) = \text{constant}$

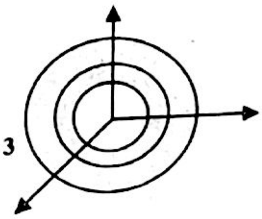


Fig. 3

Figure 3 shows the level surface for $w = x^2 + y^2 + z^2$.

Geometrical Meaning of Partial Derivative

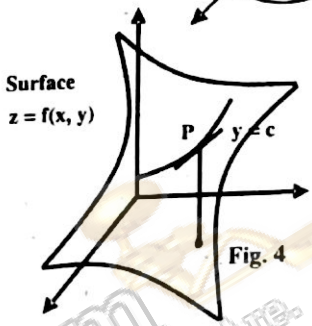


Fig. 4

Suppose $z = f(x, y)$ is a function of two variables.
We know that z represents a surface in R^3.
Now consider those points of the surface $z = f(x, y)$ for which $y = c$. Geometrically, this means we are considering those points of the surface where the surface $z = f(x, y)$ and the plane $y = c$ intersect each other which in fact is a curve. (see the figure)
On this curve z changes with x while y remained constant. Therefore, $\partial z/\partial x$ represents a slope of the tangent to this curve $z = f(x, c)$ at the point P. This is shown in the above figure.
Similarly, $\partial z/\partial y$ is the slope of the curve $z = f(d, y)$ at the point $(d, y, f(d, y))$.

Example 01: If $f(x,y) = \dfrac{x^2+y^2}{x+y}$, prove that $(f_x - f_y)^2 = 4(1 - f_x - f_y)$

Solution: Differentiating f partially w.r.t x, we get,

$$f_x = \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}.$$

Similarly by symmetry, we have $f_y = \dfrac{y^2+2xy-x^2}{(x+y)^2}$. Thus,

$$\text{LHS} = (f_x - f_y)^2 = \left[\frac{(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}\right]^2 = \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2$$

$$= 4\left[\frac{(x-y)(x+y)}{(x+y)^2}\right]^2 = 4\left[\frac{(x-y)}{(x+y)}\right]^2 \qquad (1)$$

$$\text{RHS} = 4(1 - f_x - f_y) = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= 4\frac{x^2-2xy+y^2}{(x+y)^2} = 4\left[\frac{(x-y)}{(x+y)}\right]^2 \qquad (2)$$

From (1) and (2) it follows that: $(f_x - f_y)^2 = 4(1 - f_x - f_y)$

Example 02: If $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $xu_x + yu_y = 0$.

Solution: Differentiating u partially w.r.t x, we get,

$$u_x = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow \quad xu_x = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \qquad (1)$$

Similarly, $u_y = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$

$$\Rightarrow \quad yu_x = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \qquad (2)$$

Adding (1) and (2), we get: $xu_x + yu_y = 0$

Example 03: If $u = (1-2xy+y^2)^{-1/2}$ prove that $xu_x - yu_y = y^2u^3$

Solution: Given $u = (1-2xy+y^2)^{-1/2}$. Differentiating partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = u_x = -\frac{1}{2}\left(1-2xy+y^2\right)^{-3/2}(-2y) = \frac{y}{\left(1-2xy+y^2\right)^{3/2}}$$

$$\Rightarrow \quad xu_x = xy\left(1-2xy+y^2\right)^{-3/2} \qquad (1)$$

Differentiating partially w.r.t y, we get

$$\frac{\partial u}{\partial y} = u_y = -\frac{1}{2}\left(1-2xy+y^2\right)^{-3/2}(-2x+2y) = \frac{(x+y)}{\left(1-2xy+y^2\right)^{3/2}}$$

$$\Rightarrow \quad yu_y = \left(xy-y^2\right)\left(1-2xy+y^2\right)^{-3/2} \qquad (2)$$

Now adding (1) and (2), we obtain

$$xu_x - yu_y = xy\left(1-2xy+y^2\right)^{-3/2} - \left(xy-y^2\right)\left(1-2xy+y^2\right)^{-3/2}$$

$$= \left(xy-xy+y^2\right)\left[\left(1-2xy+y^2\right)^{-1/2}\right]^3 = y^2u^3$$

Example 04: If $Z = e^{(ax+hy)} f(ax-by)$, prove that $b\,Z_x + a\,Z_y = 2ab\,Z$

Solution: Given that $Z = e^{(ax+by)} f(ax-by)$. Differentiate partially w.r.t x, we get

$Z_x = a\,e^{(ax+by)} f(ax-by) + a\,e^{(ax+by)} f'(ax-by)$

$\Rightarrow \quad bZ_x = ab\,e^{(ax+hy)} [f(ax-by) + f'(ax-by)]$ (1)

Now differentiating partially w.r.t y, we get

$Z_y = b\,e^{(ax+by)} f(ax-by) - b\,e^{(ax+by)} f'(ax-by)$

$\Rightarrow \quad aZ_y = ab\,e^{(ax+hy)} [f(ax-by) - f'(ax-by)]$ (2)

Adding (1) and (2, we get:

$bZ_x + aZ_y = ab\,e^{(ax+by)} [f(ax-by) + f'(ax-by)] + ab\,e^{(ax+by)} [f(ax-by) - f'(ax-by)]$

$= ab\,e^{(ax+hy)}[\,f(ax-by) + f'(ax-by) + f(ax-by) - f'(ax-by)]$

$= ab\,e^{(ax+by)}[\,2f(ax-by)] = 2ab\,[e^{(ax+by)} f(ax-by)] = 2abZ$

Partial Derivatives of Higher Orders

The partial derivatives f_x and f_y may possess derivatives where f is a function of two variables x and y.

In such cases, we may define the second order partial derivatives as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 f}{\partial x^2}=f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 f}{\partial y\partial x}=f_{yx} \text{ and } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 f}{\partial x\partial y}=f_{xy}$$

The partial derivatives f_{xy} and f_{yx} are called mixed derivatives. In general they are not equal, however, if they both exist (finite), then: $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$, or $f_{xy}=f_{yx}$

For example, let $z = f(x, y)= x^2 y^2$, then

$f_x = 2xy^2$, $f_y = 2x^2 y$, $f_{xx} = 2y^2$, $f_{yy} = 2x^2$, $f_{xy} = 4xy$, $f_{yx} = 4xy$.

You may observe that: $f_{xy} = f_{yx}$

Example 05: Show that $f_{xy} = f_{yx}$, if $f(x, y) = \sin^{-1}(x/y)$.

Solution: Differentiating f partially w.r.t x, we get

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=\frac{y}{\sqrt{y^2-x^2}}\frac{1}{y}=\frac{1}{\sqrt{y^2-x^2}} \quad (1)$$

Differentiating (1) partially w.r.t y now, we get

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\frac{1}{\sqrt{y^2-x^2}}=\frac{\partial}{\partial y}(y^2-x^2)^{-1/2}=-1/2(y^2-x^2)^{-3/2}(2y+0)=\frac{-y}{(y^2-x^2)^{3/2}}$$

Now differentiating f partially w.r.t y, we get (2)

$$\frac{\partial f}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)=\frac{y}{\sqrt{y^2-x^2}}\times\frac{-x}{y^2}=\frac{-x}{y\sqrt{y^2-x^2}} \quad (3)$$

Differentiating (3) w.r.t x, we obtain

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{-x}{\sqrt{y^2-x^2}}=\frac{-1}{y\sqrt{y^2-x^2}}-\frac{x}{y}\left[\frac{x}{(y^2-x^2)^{3/2}}\right]=\frac{-y}{(y^2-x^2)^{3/2}} \quad (4)$$

From (2) and (4) we see that: $f_{xy} = f_{yx}$

Example 06: If $f(x, y) = e^x \sin y + e^y \cos x$, show that f satisfies Laplace equation $f_{xx} + f_{yy} = 0$.

Solution: Differentiating f w.r.t x, we get: $f_x = e^x \sin y - e^y \sin x$

Differentiating again w.r.t x, we have: $f_{xx} = e^x \sin y - e^y \cos x$ (1)

Now differentiating f w.r.t y, we get: $f_y = e^x \cos y + e^y \cos x$

Differentiating again w.r.t y, we obtain: $f_{yy} = - e^x \sin y + e^y \cos x$ (2)

Adding (1) and (2) we get

$$f_{xx} + f_{yy} = 0$$

Example 07: If $f(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$, **show that** $\frac{\partial^2 f}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}$

Solution: Differentiating f w.r.t y, we get

$$\frac{\partial f}{\partial y}=x^2\frac{1}{\left[1+\left(\frac{y}{x}\right)^2\right]}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)-\left[y^2\frac{1}{\left(1+\frac{x}{y}\right)^2}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right).2y\right]$$

$$=x^2\frac{x^2}{\left[x^2+y^2\right]}\frac{1}{x}-\left[y^2\frac{y^2}{\left(y^2+x^2\right)}\times\left(\frac{-x}{y^2}\right)+\tan^{-1}\left(\frac{x}{y}\right).2y\right]$$

$$=\frac{x^3}{\left(y^2+x^2\right)}+\frac{xy^2}{\left(y^2+x^2\right)}-2y\tan^{-1}\left(\frac{x}{y}\right)=\frac{x\left(x^2+y^2\right)}{\left(y^2+x^2\right)}-2y\tan^{-1}\left(\frac{x}{y}\right)=x-2y\tan^{-1}\left(\frac{x}{y}\right)$$

Now differentiating w.r.t x, we get

$$\frac{\partial^2 f}{\partial x\partial y}=1-2y\frac{1}{\left[1+\frac{x}{y}\right]^2}\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=1-\frac{2y^3}{\left(y^2+x^2\right)}.\frac{1}{y}=\frac{y^2+x^2-2y^2}{y^2+x^2}=\frac{x^2-y^2}{x^2+y^2}$$

Example 08: If $U = f(x + at) + g(x - at)$, show that $U_{tt} = a^2 U_{xx}$

Solution: Differentiate U partially w.r.t x, we get

$$U_x = f'(x + at) + g'(x - at)$$

Differentiating again partially w.r.t x, we get

$$U_{xx} = f''(x + at) + g''(x - at) \quad (1)$$

Similarly differentiating twice partially w.r.t t, we get

$$U_{tt} = a^2[f''(x + at) + g''(x - at)] \quad (2)$$

From (1) and (2), we see that: $U_{tt} = a^2 U_{xx}$

Example 09: Let $V=\left(x^2+y^2+z^2\right)^{m/2}, m\neq 0$. If $V_{xx} + V_{yy} + V_{zz} = 0$ then find m.

Solution: Given that $V=\left(x^2+y^2+z^2\right)^{m/2}$. Differentiate partially w.r.t x, we get

$V_x=\frac{m}{2}\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}.2x=mx\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}$. Differentiate again w.r.t x

$$V_{xx}=m\left[\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}.1+x\frac{m-2}{2}\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}-1}.2x\right]$$

$$=m\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}\left[1+(m-2)x^2\left(x^2+y^2+z^2\right)^{-1}\right]$$

Similarly, $V_{yy}=m\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}\left[1+(m-2)y^2\left(x^2+y^2+z^2\right)^{-1}\right]$

And, $V_{zz} = m\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}\left[1+(m-2)z^2\left(x^2+y^2+z^2\right)^{-1}\right]$.

$$\Rightarrow V_{xx}+V_{yy}+V_{zz} = m\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}\left[3+(m-2)\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1}\right]=0$$

$$= m\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}}\left[3+(m-2)\right] = m(m+1)\left(x^2+y^2+z^2\right)^{\frac{m-2}{2}} = 0$$

$$\Rightarrow m(m+1)=0 \quad \Rightarrow m=0 \text{ or } m=-1$$

It is given that $m \neq 0$. Thus $m=-1$.

Example 10: Suppose that the revenue from the sale of x Model-A stereo speakers and y Model-B stereo speakers is given by

$$\mathbf{R(x, y) = 100x + 150y - 0.3x^2 - 0.02y^2 \text{ dollars}}$$

Determine the rate at which revenue will change with respect to the change in the number of model-A speakers sold, when 50 Model-A speakers and 40 Model-B speakers have been sold?

Solution: The rate of change we seek is $\frac{\partial R}{\partial x}(50,40)$.

Since, $R(x,y)=100x+150y-0.3x^2-0.02y^2$ (1)

Differentiating (1) partially with respect to x, we get

$$\frac{\partial R}{\partial x}=100-0.06x \cdot \text{At } (50, 40),\ \frac{\partial R}{\partial x}(50,40)=100-0.06(50)=97$$

This means that the additional revenue that will be obtained from the sale of Model-A speakers is approximately \$97, assuming that 50 Model-A speakers and 40 Model-B speakers have been sold.

Differentiability

Let z = f(x, y) be a function of two variables. The differential of z is defines as:

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

In case if z = f(x, y) is constant then dz = 0. In this case

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=0 \ \Rightarrow f_x dx+f_y dy=0 \ \Rightarrow \frac{dy}{dx}=-\frac{f_x}{f_y} \qquad (1)$$

Formula (1) is used to find the ordinary derivative if a function is given in implicit form.

Example 11: Find dy/dx if $x^3 + x^2y + y^3 = 0$

Solution: Here $f(x, y) = x^3 + x^2y + y^3 \rightarrow f_x = 3x^2 + 2xy$ and $f_y = x^2 + 3y^2$.

$$\rightarrow \quad \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{3x^2+2xy}{x^2+3y^2}$$

Example 12: Find dy/dx if $e^{xy} + \sin xy = 0$

Solution: Here $f(x, y) = e^{xy} + \sin xy$

$\rightarrow$ $f_x = ye^{xy} + y\cos xy$

And $f_y = xe^{xy} + x\cos xy$

Now, $$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{y\left(e^{xy}+\cos xy\right)}{x\left(e^{xy}+\cos xy\right)}=-\frac{y}{x}$$

WORKSHEET 04

1. Evaluate the following limits:

(i) $\lim\limits_{\substack{x\to 0\\ y\to 2}} \dfrac{3xy}{x^2+y^2}$ (ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x-y}{x+5y}$ (iii) $\lim\limits_{\substack{x\to 1\\ y\to \infty}} \dfrac{3xy}{x^2+y^2}$ (iv) $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{xy-2x}{xy-2y}$

2. If $f(x,y)=\dfrac{2x-y}{2x+y}$ show that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$

3. Show that the function f(x, y) defined as

$$f(x,y)=\begin{cases} 2x^2+y, & (x,y)\neq(1,2) \\ 0 & , \quad (x,y)=(1,2)\end{cases}$$

is not continuous at (x, y) = (1, 2).

4. Show that the function $f(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)$ and $f(x,y)=\tan^{-1}\left(\dfrac{2xy}{x^2-y^2}\right)$

satisfies the Laplace equation $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$

5. Show that $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$ where:

(a) $f(x, y) = \ln(e^x + e^y)$ (b) $f(x, y) = \ln(x^2 + y^2) - \ln(xy)$

(c) $f(x, y) = x \sin xy + y \cos xy$ (d) $f(x, y) = \ln(e^x + e^y)$

6. If $Z = \ln(e^x + e^y)$, show that $Z_{xx} \cdot Z_{yy} = (Z_{xy})^2$

7. If $u = r^m$ and $r^2 = x^2 + y^2 + z^2$, show that $u_{xx} + u_{yy} + u_{zz} = m(m + 1)\, r^{m-2}$

8. If $ux + vy = 0$ and $u/x + v/y = 1$, show that $u_x - v_y = [(x^2 + y^2)/(y^2 - x^2)]$

9. If $u = x^y$, prove that $\dfrac{\partial^3 u}{\partial x^2\partial y}=\dfrac{\partial^3 u}{\partial x\partial y\partial x}$

10. Find dy/dx in each of the following case. Verify the results by using the formula $dy/dx = -f_x/f_y$.

(a) $y^2 + x^2y + a x^4 = 0$ (b) $x^3 + x^2 + xy^2 + \sin y = 0$

CHAPTER FIVE

FUNDAMENTAL THEOREMS AND INDETERMINATE FORMS

In this chapter we shall discuss some important theorems of fundamental importance in calculus. These theorems help to study the behavior of various functions. We shall also study an indeterminate forms and a very important topic called "Asymptotes".

5.1 ROLLE'S THEOREM

Statement: If f(x) is a function such that

(i) It is continuous on the closed interval [a, b]

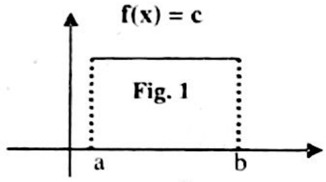
(ii) It is derivable in the open interval (a, b)

(iii) f(a) = f(b)

then there exists at least one value 'c' in the open interval (a, b) such that $f'(c) = 0$.

Proof: Since given function f(x) is continuous on [a, b] hence it is bounded therein and attains its bound. Let M and m be the upper and lower bounds of f on [a, b]. There occur two cases:

Case I: When M = m, that is the upper and lower bounds are equal. In this case the function f(x) is constant (see figure 1) and so its derivative at every point in (a, b) is zero, that is, f(x) = c (constant) then $f'(x) = 0$ for $x \in (a, b)$. Hence the theorem is true in this case. See figure 1.



If $M \neq m$, then at least one of them will be different from f(a) and f(b). This is shown in figure 2.

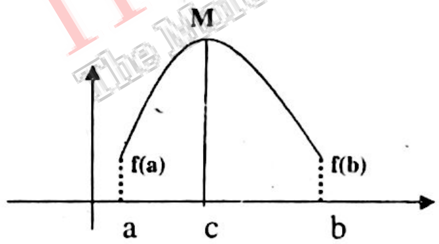
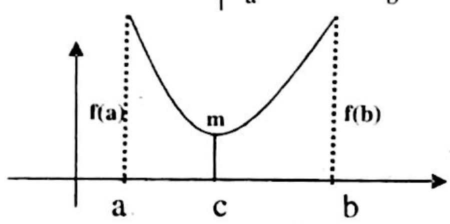


Fig. 2



Now suppose that: $M \neq f(a) = f(b)$ (1)

Let the point where the upper bound M occur be c that is, f(c) = M. Then from (1) 'c' must be different from a and b. This implies that 'c' lies inside the interval [a, b], that is $c \in (a, b)$. Take 'h' as a positive real number such that c − h and c + h both lie in the interval (a, b). Then: $f(c - h) \leq f(c)$ and $f(c + h) \leq f(c)$

$$\Rightarrow \quad \frac{f(c-h)-f(c)}{h} \leq 0 \qquad (2)$$

and
$$\frac{f(c-h)-f(c)}{h} = \frac{f(c+(-h))-f(c)}{h} \leq 0$$

Multiplying both sides by -1, we get: $\frac{f(c+(-h))-f(c)}{-h} \geq 0$ (3)

Taking limit h ➔ 0, we obtain respectively from (2) and (3)

$f'(c) \le 0$ and $f'(c) \ge 0$. This implies that: $f'(c) = 0$.

This proves the theorem.

Geometrical Interpretation of Rolle's Theorem

Since f(a) = f(b) the ends of the graph of f(x) are at the same horizontal level. Since f(x) is continuous, the graph of f(x) is either a horizontal line or a smooth curve joining the points $(a, f(a))$ and $(b, f(b))$.

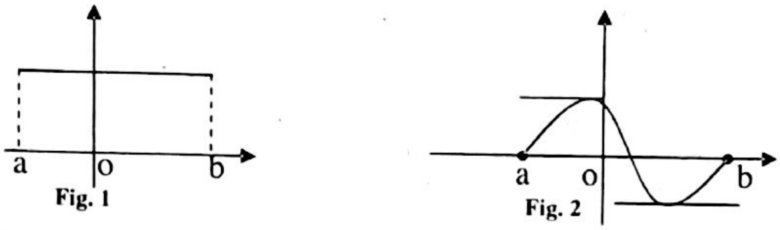


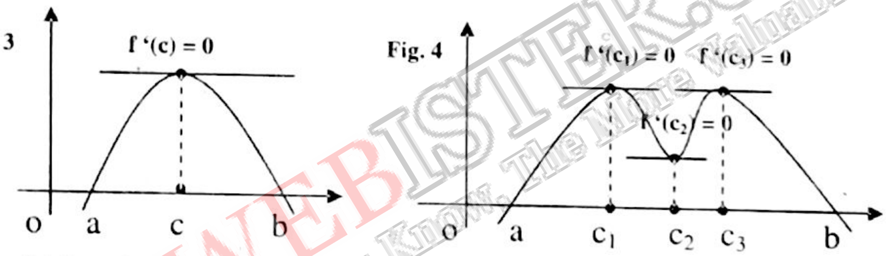
Fig. 1 Fig. 2

In the first case the slope of a function is zero for all x. In the latter case, the graph must have turning points hence the tangent at these points must be horizontal. In other words, the slope of the tangent at such points is always zero.

Remarks: A 300–year old theorem of Michel Rolle (1652 – 1719) assures that, if any function satisfies all three conditions as mentioned above, then there exists at least one point 'c' in (a, b) where the slope of the tangent line at x = c is zero, which means this tangent line is parallel to the x-axis.

Other two graph are also shown here.

Fig. 3 Fig. 4



Physical Meaning of Rolle's Theorem

Let a stone be thrown from the ground into the air. Suppose the height of the stone after time t be s = f(t). Surely the stone will hit the ground after some time so, f(0) = f(T) = 0. The function s = f(t) satisfies the conditions of Roll's theorem on the interval [0, T]. Hence at certain time $t_0 \in (0, T)$ the velocity of the stone is zero, that is; $f'(t_0) = 0$. We know that indeed it happens.

It may be noted that Rolle's Theorem does not guarantee to hold for a function that does not satisfy any one of the three conditions given above. There may or may not occur such point.

REMARKS:

(i) Every polynomial and e^x, sin x, cos x are continuous for all real x.

(ii) log x is a continuous function for all $x > 0$.

(iii) If f and g are both derivable on the closed interval [a, b] then $f \pm g$ and fg are also derivable on [a, b]. f/g is derivable in (a, b), provided $g(x) \neq 0$ for any $x \in (a, b)$.

Example 01: Verify Rolle's Theorem in each of the following cases for
(a) $f(x) = x^2 - 6x + 8$ on [2, 4].

Solution: Here a = 2, and b = 4.

(i) Since f(x) is a polynomial, therefore, it is continuous on [2, 4].

(ii) $f'(x) = 2x - 6$, which exists (finite) in the open interval (2, 4)

(iii) $f(2) = 4 - 12 + 8 = 0, f(4) = 16 - 24 + 8 = 0 \Rightarrow f(2) = f(4) = 0$

Hence, $f(x) = x^2 - 6x + 8$ satisfies all three conditions of Rolle's Theorem. So there must exist at least one number c between 2 and 4 such that $f'(c) = 0$. Now $f'(x) = 2x - 6 \Rightarrow f'(c) = 2c - 6 = 0 \Rightarrow c = 3$. This is a point in the open interval (2, 4) and thus, the theorem is verified.

(b) $f(x) = 1 - x^{2/3}$ on [-1, 1]

Solution: (i) The given function is not defined for any negative value of x. In other words the given function is not continuous in the interval [-1, 1].

(ii) Also $f'(x) = 0 - \frac{2}{3}x^{-1/3} = \frac{-2}{3\sqrt[3]{x}}$, which does not exist at $x = 0 \in (-1, 1)$ and so f is not differentiable. Hence Rolle's Theorem is not applicable to the given function on [-1, 1].

(c) $f(x) = x(x + 3)e^{-x/2}$ on [-3, 0]

Solution: (i) Since $x(x+3) = x^2 + 3x$ and $e^{-x/2}$ are continuous functions for all x, therefore their product $x(x+3)e^{-x/2} = (x^2 + 3x)e^{-x/2}$ is also a continuous for all x. It implies that f (x) is continuous in [-3, 0].

(ii) $f'(x) = -\frac{(x^2+3x)e^{-x/2}}{2} + e^{-x/2}(2x+3) = (2x+3)e^{-x/2} - \frac{1}{2}(x^2+3x)e^{-x/2}$

$$= -\frac{1}{2}\left[-2(2x+3) + x^2 + 3x\right]e^{-x/2} = -\frac{1}{2}(-4x - 6 + x^2 + 3x)e^{-x/2}$$

$f'(x) = -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$, which is derivable (finite) in (-3, 0) and hence $f'(x)$ exists.

(iii) $f(a) = f(-3) = -3(-3+3)e^{-3/2} = 0, f(b) = f(0) = 0 \Rightarrow f(a) = f(b) = 0$.

Thus, f satisfies all three conditions of Roll's theorem. So, there must exist at least one number c in (-3, 0) such that $f'(c) = 0$. That is;

$$f'(c) = -\frac{1}{2}(c^2 - c - 6)e^{-c/2} = 0 \qquad \Rightarrow (c^2 - c - 6) = 0$$

➔ $c = -2, 3$. [Note: $e^x \neq 0$ for any finite value of x]

But $c = -2 \in (-3, 0)$ whereas $c = 3 \notin (-3, 0)$. Hence Rolle's Theorem is valid and c = -2.

(d) $f(x) = \sin x / e^x$ in $(0, \pi)$

Solution: Since both sin x and e^x are continuous in $(0, \pi)$ and derivable on $[0, \pi]$, hence there exists a point c in side $(0, \pi)$ such that $f'(c) = 0$.

Now $f'(x) = \frac{e^x \cos x - \sin x e^x}{(e^x)^2} = \frac{\sin x - \cos x}{e^x}$

➔ $f'(c) = \frac{\sin c - \cos c}{e^c} = 0$ ➔ $\sin c - \cos c = 0$ ➔ $\sin c = \cos c$ ➔ $\sin c / \cos c = 1$

➔ $\tan c = 1$ ➔ $c = \pi/4$. We know that $\pi/4$ belongs to $(0, \pi)$.

This verifies Roll's Theorem with $c = \pi/4$.

(e) $f(x) = (x - a)^m (x - b)^n$ on [a, b] where m, n are positive integers and a, b are real.

Solution: Since m and n are positive integers so given function is product of two polynomials. Hence it is continuous as well as derivable every where. Now

$$f'(x) = (x-b)^n . m(x-a)^{m-1} + (x-a)^m . n(x-b)^{n-1}$$

$$= (x-a)^{m-1}(x-n)^{n-1}\left[m(x-b)+n(x-a)\right]$$

$$\Rightarrow f'(c) = (c-a)^{m-1}(c-n)^{n-1}\left[m(c-b)+n(c-a)\right] = 0$$

$$\Rightarrow \left[m(c-b)+n(c-a)\right] = 0 \Rightarrow mc - mb + nc - na = 0$$

$$\rightarrow c(m+n) = (na+mb) \rightarrow c = \frac{(na+mb)}{(m+n)}$$

Mean Value Theorem

Statement: If a function f is such that

(i) It is continuous on the closed interval [a, b]

(ii) It is derivable in the open interval (a, b)

then there exists at least one value 'c' in the open interval (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof: Let us define a function G(x) by

$$G(x) = A x + f(x)$$

where A as some constant to be determined such that G(a) = G(b).

Since Ax is a continuous as well as derivable function for all real x, hence G(x) is also continuous and derivable function. Thus G(x) must satisfy all three conditions of Rolle's Theorem including $G`(c) = 0$. Now

$G`(x) = A + f`(x) \rightarrow G`(c) = A + f`(c) \rightarrow A + f`(c) = 0 \rightarrow A = - f`(c).$

Now, $G(a) = G(b) \rightarrow A a + f(a) = A b + f(b) \rightarrow A = [f(a) - f(b)]/(b - a)$

$\rightarrow$ $-f`(c) = [f(a) - f(b)]/(b - a)$

Or $$\frac{f(b)-f(a)}{b-a} = f'(c)$$

This verifies the "Mean Value Theorem".

Geometrical Meaning of Mean Value Theorem

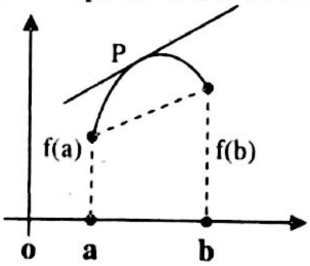
Let A and B be points on the graph of the function $y = f(x)$ corresponding to $x = a$ and $x = b$. Therefore the coordinates of the points A and B are (a, f(a)) and (b, f(b)) respectively. Hence,

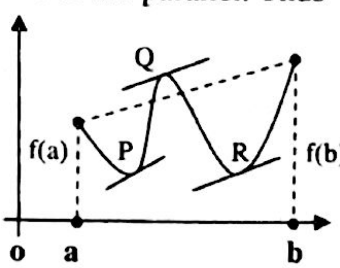
$$\text{Slope of chord AB} = \frac{\text{difference of ordinates}}{\text{difference of abscissae}} = \frac{f(b)-f(a)}{b-a}.$$

Now, MVT states that if any function satisfies two conditioned as mentioned above, then there exist at least one point c inside the interval (a, b) where the slopes of tangent and chord are equal. In other words at such point chord and tangent lines are parallel. Thus

$$\frac{f(b)-f(a)}{b-a} = f'(c),$$

This is shown in the adjacent figure.





Example 02: Verify Mean Value Theorem for the following function and find c if possible.

(i) $f(x) = (x-1)(x-2)(x-3)$ on [0, 4]

Solution: Here

$$f(x)=(x-1)(x-2)(x-3)=(x^2-3x+2)(x-3)=x^3-6x^2+11x-6$$

is a polynomial hence it is continuous as well as derivable on the interval [0, 4].

Also, $f'(x)=3x^2-12x+11$

Therefore, by the Mean Value Theorem, we have

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2-12c+11=\frac{f(4)-f(0)}{4-0}$$

➔ $$3c^2-12c+11=\frac{(64-96+44-6)-(-6)}{4}$$

➔ $$3c^2-12c+11=3 \Rightarrow 3c^2-12c+8=0$$

Using quadratic formula, we get

$$c=2+1.155, \text{ and } c=2-1.155 \text{ or } c=3.155 \text{ and } c=0.845$$

We observe that both values of c are admissible since both lie in (0, 4).

(ii) $f(x)=e^x$ on [0, 1].

Solution: $f(x) = e^x$ is continuous as well as derivable function. Now,

$f(0) = e^0 = 1$, $f(1) = e$. Also $f'(x)=e^x \Rightarrow f'(c)=e^c$

Thus by MVT, $f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow e^c=\frac{e-1}{1-0}=e-1 \Rightarrow \ln e^c=\ln(e-1)$

$\Rightarrow c=\ln(e-1)=0.541 \in (0, 1)$. This verifies MVT.

5.2 INFINITE SERIES

A question that frequently arises in both engineering and mathematical problem-solving is the behavior of a solution when one (or more) parameters in the problem statement are changed. This occurs in sensitivity analysis when we examine solutions for their dependence on errors in the original data. One of the mathematical tools for such analysis is Taylor's Theorem. In this section we shall develop the theorem and then use it to solve problems.

Maclaurin's Series

Suppose the function f (x) is represented within a certain interval (including x = 0) by a power series of the form:

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_{n-1}x^{n-1}+\cdots$$

where a's are constants.

Now differentiating the power series term by term, w.r.t x, we get

$$f'(x)=1a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots$$

$$f''(x)=0+1.2a_2+2.3a_3x+4\cdot 3a_4x^2+\cdots$$

$$f'''(x)=2.3a_3+2.3.4a_4x+\cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$f^{(n-1)}(x)=(n-1)!a_{n-1}+n!a_nx+\cdots$$

Assuming that derivatives of all orders exist at x = 0, then

$$f'(0)=a_1 \Rightarrow a_1=f'(0), f''(0)=2!a_2 \Rightarrow a_2=\frac{f''(0)}{2!}, f'''(0)=3!a_3 \Rightarrow a_3=\frac{f'''(0)}{3!},\ldots$$

$$f^{(n-1)}(0)=(n-1)!a_{n-1} \Rightarrow a_{n-1}=\frac{f^{(n-1)}(0)}{(n-1)!},\ldots$$

Thus, power series becomes

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}f^n(0)$$

This is known as Maclaurin's series.

It should be noted that a function cannot be represented by a Maclaurin's series unless the function and all its derivatives exist at x = 0. Maclaurin's series is useful in computing the value of a function only when x is small (near to zero).

Example 01: Find the Maclaurin's series of the following functions:
(i) sin x (ii) e^x.

Solution: (i) Given $f(x)=\sin x$

Differentiating successively with respect to x, we get

$$f'(x)=\cos x, f''(x)=-\sin x, f'''(x)=-\cos x, f^{(iv)}(x)=\sin x,$$
$$f^{(5)}(x)=\cos x, f^{(6)}(x)=-\sin x, f^{(7)}(x)=-\cos x, f^{(8)}(x)=\sin x,\cdots$$

Substituting x = 0, we get

$$f(0)=0, f'(0)=1, f''(0)=0, f'''(0)=-1, f^{(4)}(0)=0,$$
$$f^{(5)}(0)=1, f^{(6)}(0)=0, f^{(7)}(0)=-1, f^{(8)}(0)=0,\ldots$$

The Maclaurin's series is

$$f(x)=f(0)+xf'(0)+x^2\frac{f''(0)}{2!}+x^3\frac{f'''(0)}{3!}+x^4\frac{f^{(4)}(0)}{4!}+\ldots$$

$$\therefore \sin x=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-1)++\frac{x^4}{4!}(0)+\frac{x^5}{5!}(1)+\frac{x^6}{6!}(0)+\frac{x^7}{7!}(-1)+\frac{x^8}{8!}(0)+\ldots$$

Thus, $$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

(ii) Given $f(x)=e^x$

Differentiating successively with respect to x, we get

$$f'(x)=e^x, f'(x)=e^x, f''(x)=e^x, f'''(x)=e^x, f^{(4)}(x)=e^x,\cdots$$

At x = 0, we get

$$f(0)=e^0=1, f'(0)=e^0=1, f''(0)=e^0=1, f'''(0)=e^0=1, f^{(4)}(0)=e^0=1,\ldots$$

The Maclaurin's series is:

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$$

Having substituted the values, we get

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

Example 02: Prove that, $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$.

Hence find the value of π.

Solution: Let $f(x)=\tan^{-1}x \rightarrow f(0)=\tan^{-1}(0)=0$

$$f'(x)=1/(1+x^2) \quad \Rightarrow f'(0)=1$$

$$f'(x)=\left(1+x^2\right)^{-1} \Rightarrow f''(x)=-1\left(1+x^2\right)^{-2}(2x) \qquad \Rightarrow f''(0)=0$$

$$f'''(x)=-2\left[\left(1+x^2\right)^{-2}(1)+x(-2)\left(1+x^2\right)^{-3}(2x)\right]=-2\left[\left(1+x^2\right)^{-2}-4x^2\left(1+x^2\right)^{-3}\right]$$

$$=8x^2\left(1+x^2\right)^{-3}-2\left(1+x^2\right)^{-2} \qquad \Rightarrow f'''(0)=-2,$$

$$f^{(4)}(x)=8\left\{x^2(-3)\left(1+x^2\right)^{-4}(2x)+\left(1+x^2\right)^{-3}(2x)\right\}-2\left\{-2\left(1+x^2\right)^{-3}(2x)\right\}$$

$$=8\left\{-6x^3\left(1+x^2\right)^{-4}+2x\left(1+x^2\right)^{-3}\right\}+8x\left(1+x^2\right)^{-3}$$

$$=-48x^3\left(1+x^2\right)^{-4}+16x\left(1+x^2\right)^{-3}+8x\left(1+x^2\right)^{-3}$$

$$\rightarrow f^{(4)}(x)=24x\left(1+x^2\right)^{-3}-48x^3\left(1+x^2\right)^{-4}, f^{(4)}(0)=0,\ldots$$

Now the Maclaurin's series is:

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$$

$$\tan^{-1}x=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-2)+\frac{x^4}{4!}(0)+\frac{x^5}{5!}(24)+\ldots$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$$

Putting $x = 1$ in the above expansion, we get

$$\tan^{-1}(1)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \Rightarrow \pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right).$$

Hence the value of π can be obtained to any degree of accuracy using the above result. It may be noted that humanely it is not possible to take infinite number of terms of this series if a value of given function is to be calculated at $x = 0$. If we take only finite number of terms of the Maclaurin's series there must occur, some error. This error part of the series is known as "Remainder". The remainder of the Maclaurin's series is given by:

$$R_n=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x), \text{ where } 0<\theta<1$$

Thus if a function f(x) is expressed in an infinite series at $x = 0$, then it can be expressed as: $f(x) = P_n(x) + R_n(x)$

where, $P_n(x)=f(0)+f'(0)x+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots+\frac{x^n}{n!}f^n(0)$

is a polynomial of degree n and $R_n(x)$ is a remainder after (n + 1) terms.

The remainder $R_n(x)$ helps us:

(i) To compute the number of terms of the series we must take if the error is known.

(ii) To compute the error if the number of terms of a series are known.

Example 03: Find the approximate value of e by using Maclaurin's series expansion of $f(x) = e^x$ if only 10 terms of the series are taken. Show the validity of your results by taking the value of e to 27 decimal places from the window calculator.

Solution: We know that series expansion of e^x is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad (1)$$

The remainder is: $R_n = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x)$, where $0 < \theta < 1$

Now, $f(x) = e^x \rightarrow$ $f^{(n+1)}(x) = e^x \rightarrow f^{(n+1)}(\theta x) = e^{\theta x}$

Hence, $R_n = \dfrac{x^{n+1}}{(n+1)!} e^{(\theta x)}$, where $0 < \theta < 1$

Since $x = 1$, and $0 < \theta < 1$, hence $x^{n+1} e^{\theta x} \sim 1$.

Thus $R_n \sim 1/(n+1)!$.

Now if $n = 10$, then $R_n \sim 1/11! = 0.00000002$. This shows that if 10 terms of the series (1) are taken and if we take $x = 1$ to find the value of e, there will be an error of about 0.00000002 (7 dp).

Now take 10 terms of the series (1) and putting $x = 1$, we get

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \sim 2.718281526 \qquad (2)$$

Now the value of e to 27 decimal places (from the window calculator) is

$$e = 2.718284590452353602874713527 \qquad (3)$$

This agrees with the above result in (2) to 5 decimal places.

Although our result should have been accurate to 7 decimal places as is clear from the value of R_n, the difference between the values of e in equations (2) and (3) is due to the approximate value of $e^{\theta x}$ which we have assumed to be equal to 1 whereas it isn't.

Taylor's Series

Unless a function is defined at $x = 0$ and all of its derivatives also exist at $x = 0$, the function cannot be represented by a Maclaurin's series.

Functions such as $x^{5/3}, \ln x, \cot x$ and $\csc x$ cannot be represented by a Maclaurin's series.

Let $a \in R$ be different from zero and suppose that $f(x)$ is a function which is represented within a certain interval by a power series of the form shown below in which c's are constants and $f(a) = c_0$ · Differentiating term by term with respect to x, we get

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots + c_{n-1}(x-a)^{n-1} + \dots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

$$f''(x) = 2!c_2 + 3!c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \dots$$

$$f'''(x) = 3!c_3 + 4!c_4(x-a) + \dots$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$f^{(n-1)}(x) = (n-1)!c_{n-1} + n!\,c_n(x-a) + \cdots$$

Since derivatives of all orders exist at $x = a$, we get

$$f'(a) = c_1 \Rightarrow c_1 = f'(a), f''(a) = 2!c_2 \Rightarrow c_2 = \frac{f''(a)}{2!}, f'''(a) = 3!c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}, \dots$$

$$f^{(n-1)}(a) = (n-1)!c_{n-1} \Rightarrow c_{n-1} = \frac{f^{(n-1)}(a)}{(n-1)!}, \dots$$

Thus, the power series becomes

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\ldots$$

This is called the 'Taylor's series.' It is also known as the series expansion or development of $f(x)$ in powers of $(x-a)$. The remainder is

$$R_n=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta x),\text{ where } 0<\theta<1$$

Another useful form of Taylor's series can be obtained if we replace x by a + h that is;

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)\ldots$$

Note that both forms of Taylor's expansion reduce to Maclaurin's expansion if a = 0. Thus Maclaurin's series becomes a special case of Taylor's series.

Example 04: Obtain the Taylor's development of the following function f(x) = ln x at a = 1 up to 5 terms. Also compute ln 2.

Solution: Given that

$$f(x)=\ln x\Rightarrow f(a)=f(1)=\ln(1)=0,\ f'(x)=\frac{1}{x}\Rightarrow f'(a)=f'(1)=1/1=1,$$

$$f''(x)=-\frac{1}{x^2}\Rightarrow f''(a)=f''(1)=-1,\ f'''(x)=\frac{2}{x^3}\Rightarrow f'''(a)=f'''(1)=2$$

$$f^{(4)}(x)=\frac{-6}{x^4}\Rightarrow f^{(4)}(a)=f^{(4)}(1)=-6,\ldots$$ The Taylor's series is:

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots$$

Substituting a = 1, we get

$$f(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4+\ldots$$

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}-\ldots$$

Replacing x by $x+1$, we get: $\log(x+1)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

At $x=1$, $\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

Example 05: Apply Taylor's theorem to prove that

$$(i)\ a^{x+h}=a^x\left[1+h\log a+\frac{h^2}{2!}(\log a)^2+\frac{h^3}{3!}(\log a)^3+\ldots\right]$$

$$(ii)\ \log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}+\ldots$$

$$(iii)\ \ln\sin(x+h)=\ln\sin x+h\cot x-\frac{h^2}{2}\csc^2 x+\frac{h^3}{3}\cot x\csc^2 x-\ldots$$

Solution: (i) Let $f(x+h)=a^{x+h}$.

Putting $h=0$, we have $f(x)=a^x$

$$f'(x)=a^x\cdot\log a, f''(x)=a^x(\log a)^2, f'''(x)=a^x(\log a)^3,\ldots$$

Hence, $f(x+h)=a^{x+h}=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$

$$=a^{x+h}=a^x+ha^x(\log a)+\frac{h^2}{2!}a^x(\log a)^2+\frac{h^3}{3!}a^x(\log a)^3+\ldots$$

$\Rightarrow\quad f(x+h)=a^{x+h}=a^x\left[1+h(\log a)+\frac{h^2}{2!}(\log a)^2+\frac{h^3}{3!}(\log a)^3+\ldots\right]$

(ii) Let $f(x+h)=\log(x+h)$

Putting $h=0$, we have $f(x)=\log x$

$$f'(x)=\frac{1}{x}, f''(x)=-\frac{1}{x^2}, f'''(x)=\frac{2}{x^3},\ldots$$

Hence, $\log(x+h)=f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$

$$=f(x+h)=\log x+h\left(\frac{1}{x}\right)+\frac{h^2}{2}\left(-\frac{1}{x^2}\right)+\frac{h^3}{6}\left(\frac{2}{x^3}\right)+\ldots$$

$\Rightarrow\quad \log(x+h)=f(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\ldots$

(iii) Let $f(x+h)=\ln\sin(x+h)$

Putting $h=0$ we have $f(x)=\ln\sin x$

$f'(x)=\frac{1}{\sin x}(\cos x)=\cot x, f''(x)=-\csc^2 x, f'''(x)=-2\csc x(-\csc x\cot x)=2\cot x\csc^2 x,\ldots$

Hence, $\ln\sin(x+h)=f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$

$$=\ln\sin x+h\cot x+\frac{h^2}{2}\left(-\csc^2 x\right)+\frac{h^3}{6}\left(2\cot x\csc^2 x\right)+\ldots$$

$\Rightarrow\quad \ln\sin(x+h)=f(x+h)=\ln\sin x+h\cot x-\frac{h^2}{2}\csc^2 x+\frac{h^3}{3}\cot x\csc^2 x-\ldots$

5.3 L'HOPITAL'S RULE AND INDETERMINATE FORMS

In the late seventeenth century, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approached zero. The rule is known today as "L, Hopital's Rule", after Guillaume Francois Antoine de L' Hopital's (1661–1704).

If the functions f(x) and g(x) are both zero at x = a, then $\lim_{x\to a} f(x)/g(x)$ cannot be found by substituting x = a. The expression produces 0/0 is meaningless expression known as an indeterminate form. The L` Hopital's Rule enabled us to evaluate such limits that lead to indeterminate forms. There are different indeterminate forms, for example: 0/0, ∞/∞, 0 × ∞, ∞ - ∞, 0^0, 1^∞, ∞^0. All these forms are called indeterminate quantities because they are undefined, indefinite, unfixed, imprecise, uncertain, and vague.

Indeterminate Form 0/0

If f and g are two functions of x which can be expanded by Taylor's series and if

$f(a)=g(a)=0$, then: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=l$

provided the latter limit exists, whether finite or infinite.

Proof: By Taylor series

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots \quad \text{(i)}$$

If we put x – a = h ➔ x = a + h, then (i) becomes

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots \quad \text{(ii)}$$

If we assume that h is so small that its square and higher powers are neglected, we get:

$$f(a+h)\sim f(a)+hf'(a)$$

But f(a) = 0, hence: $f(a+h)\sim hf'(a)$ (iii)

Similarly, $g(a+h)\sim hg'(a)$ (iv)

Dividing (iii) by (iv) and taking limit h tends to zero or x tends to a, we get

$$\lim_{h\to 0}\frac{f(a+h)}{g(a+h)}=\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{hf'(x)}{hg'(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

This proves the result.

REMARK: In case if f `(a) = 0 = g`(a), we continue applying the L'Hopital Rule till some definite value is found.

Example 01: Evaluate the following limits:

$$(i)\ \lim_{x\to 0}\frac{x-\sin x}{x^2} \quad (ii)\ \lim_{x\to 0}\frac{e^{x^2}-1}{\cos x-1}.$$

Solution: (i) $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{x-\sin x}{x^2} \quad \left(\frac{0}{0}\right)$

Using L'Hopital's Rule, we get

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{1-\cos x}{2x} \quad \left(\frac{0}{0}\right)$$

Again using L'Hopital's Rule, we get

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{\sin x}{2}=\frac{1}{2}\lim_{x\to 0}(\sin x)=0.$$

(ii) $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{e^{x^2}-1}{\cos x-1} \quad \left(\frac{0}{0}\right)$

Using L'Hopital's Rule, we get

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{2xe^{x^2}}{-\sin x} \quad \left(\frac{0}{0}\right)$$

Again using L'Hopital's Rule, we get

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{2\left\{x\left(2xe^{x^2}\right)+e^{x^2}(1)\right\}}{-\cos x}=\lim_{x\to 0}\frac{4x^2e^{x^2}+2e^{x^2}}{-\cos x}=\frac{0+2}{-1}=-2.$$

Indeterminate Form ∞/∞

If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then

$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the latter limit exists, whether finite or infinite.

Note: While evaluating $\lim_{x\to a} f(x)/g(x)$ when it is of the form ∞/∞, care must be taken to change over to the form 0/0 as soon as it is conveniently possible, otherwise the process of differentiating the numerator and denominator would never terminate.

Example 02: Evaluate $\lim_{x\to 0}\frac{\ln(\sin 3x)}{\ln(\sin x)}$.

Solution: $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\ln(\sin 3x)}{\ln(\sin x)}\quad\left(\frac{\infty}{\infty}\right)$

Using L'Hopital's Rule, we get

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\frac{1}{\sin 3x}(\cos 3x)(3)}{\frac{1}{\sin x}(\cos x)}=3\lim_{x\to 0}\frac{\sin x\cos 3x}{\sin 3x\cos x}\quad\left(\frac{0}{0}\right)$$

Again using L'Hopital's Rule, we get

$$\lim_{x\to 0} f(x)=3\lim_{x\to 0}\frac{\{\sin x(-\sin 3x)(3)+\cos 3x(\cos x)\}}{\{\sin 3x(-\sin x)+\cos x(\cos 3x)(3)\}}$$

$$=3\lim_{x\to 0}\frac{\cos 3x\cos x-3\sin 3x\sin x}{3\cos 3x\cos x-\sin 3x\sin x}=3\left(\frac{1}{3}\right)=1.$$

Indeterminate Form 0 × ∞

Let $\lim_{x\to a} f(x)=0, \lim_{x\to a} g(x)=\infty$. We write

$$f(x)\cdot g(x)=\frac{g(x)}{1/f(x)}\quad\left(\frac{\infty}{\infty}\text{ form}\right),\quad f(x)\cdot g(x)=\frac{f(x)}{1/g(x)}\quad\left(\frac{0}{0}\text{ form}\right)$$

Thus, $0\times\infty$ is changed into the form ∞/∞ or $0/0$ and then L'Hopital's Rule is applied to evaluate such limits.

Example 03: Evaluate $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$.

Solution: $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)\quad(0\times\infty)$

$$=\lim_{x\to 1}\frac{1-x}{\cot\left(\frac{\pi x}{2}\right)}\quad\left(\frac{0}{0}\right)$$

Using L'Hopital's Rule, we get

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{-1}{-\csc^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)}=\left(\frac{2}{\pi}\right)\lim_{x\to 1}\left\{\sin^2\left(\frac{\pi x}{2}\right)\right\}=\left(\frac{2}{\pi}\right)(1)=\frac{2}{\pi}.$$

Indeterminate Form $\infty - \infty$

To evaluate $\lim\limits_{x\to a}\{f(x)-g(x)\}$, when $\lim\limits_{x\to a} f(x)=\infty=\lim\limits_{x\to a} g(x)$, we have

$$\lim_{x\to a}\{f(x)-g(x)\}=\lim_{x\to a}\left\{\frac{1}{1/f(x)}-\frac{1}{1/g(x)}\right\}=\lim_{x\to a}\frac{\dfrac{1}{g(x)}-\dfrac{1}{f(x)}}{\dfrac{1}{f(x)}\cdot\dfrac{1}{g(x)}} \qquad \left(\frac{0}{0}\text{ form}\right).$$

Now, we can apply the L'Hopital's Rule.

Example 04: Evaluate the limit $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

Solution: $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) \qquad (\infty-\infty)$

$$=\lim_{x\to 1}\left[\frac{x\ln x-(x-1)}{(x-1)\ln x}\right] \qquad \left(\frac{0}{0}\right)$$

Using L'Hopital's Rule, we get

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x\left(\dfrac{1}{x}\right)+\ln x-1}{(x-1)\left(\dfrac{1}{x}\right)+\ln x(1)}=\lim_{x\to 1}\frac{\ln x}{\dfrac{(x-1)+x\ln x}{x}}=\lim_{x\to 1}\frac{x\ln x}{(x-1)+x\ln x} \qquad \left(\frac{0}{0}\right)$$

Again using L'Hopital's Rule, we get

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x\left(\dfrac{1}{x}\right)+\ln x}{1+x\left(\dfrac{1}{x}\right)+\ln x(1)}=\lim_{x\to 1}\frac{1+\ln x}{2+\ln x}=\frac{1}{2}.$$

Indeterminate Forms $0^0, 1^\infty, \infty^0$

To evaluate $\lim\limits_{x\to a}\left[\{f(x)\}^{g(x)}\right]$ when

(i) $\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$. In this case $\lim\limits_{x\to a}\left[\{f(x)\}^{g(x)}\right]$ is of the form 0^0.

(ii) $\lim\limits_{x\to a} f(x)=1$, $\lim\limits_{x\to a} g(x)=\infty$. In this case $\lim\limits_{x\to a}\left[\{f(x)\}^{g(x)}\right]$ is of the form 1^∞

(iii) $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} g(x)=0$. In this case $\lim\limits_{x\to a}\left[\{f(x)\}^{g(x)}\right]$ is of the form ∞^0.

Example 05: Evaluate the following:

(i) $\lim\limits_{x\to 0}(\tan x)^{\sin 2x}$ (ii) $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}$ (iii) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$.

Solution: (i) Let $y=(\tan x)^{\sin 2x} \Rightarrow \lim\limits_{x\to 0} y=\lim\limits_{x\to 0}(\tan x)^{\sin 2x} \qquad (0^0)$

Taking ln on both sides, we get

$$\ln\left(\lim_{x\to 0} y\right)=\lim_{x\to 0}\{\sin 2x\ln(\tan x)\} \qquad (0\times\infty)$$

$$\ln\left(\lim_{x\to 0} y\right)=\lim_{x\to 0}\frac{\ln(\tan x)}{\csc 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

Using L'Hopital's Rule, we get

$$\ln\left(\lim_{x\to0} y\right) = \lim_{x\to0}\frac{\frac{1}{\tan x}\left(\sec^2 x\right)}{-\csc 2x\cot 2x\,(2)} = \lim_{x\to0}\frac{\frac{\cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right)}{-2\csc 2x\cot 2x}$$

$$= \lim_{x\to0}\frac{1}{-2\sin x\cos x\csc 2x\cot 2x} = \lim_{x\to0}\frac{\sin 2x\tan 2x}{-\sin 2x} = \lim_{x\to0}(-\tan 2x) = 0$$

or, $\lim_{x\to0}(\tan x)^{\sin 2x} = e^0 = 1$. [This is done by taking antilog on both sides]

(ii) Let $y = (1+\sin x)^{\cot x} \Rightarrow \lim_{x\to0} y = \lim_{x\to0}(1+\sin x)^{\cot x} \quad (1^\infty)$.

Taking ln on both sides, we get

$$\ln\left(\lim_{x\to0} y\right) = \lim_{x\to0}\{\cot x\ln(1+\sin x)\} \qquad (\infty\times0)$$

$$= \lim_{x\to0}\frac{\ln(1+\sin x)}{\tan x} \qquad \left(\frac{0}{0}\right)$$

Using L'Hopital's Rule, we get

$$\ln\left(\lim_{x\to0} y\right) = \lim_{x\to0}\frac{\frac{1}{1+\sin x}(\cos x)}{\sec^2 x} = \lim_{x\to0}\frac{\cos x}{\sec^2 x(1+\sin x)} = \lim_{x\to0}\frac{\cos^3 x}{1+\sin x} = 1$$

Thus, $\lim_{x\to0}(1+\sin x)^{\cot x} = e^1 = e$. [This is done by taking antilog on both sides]

(iii) Let $y = \left(\frac{1}{x}\right)^{\tan x} \Rightarrow \lim_{x\to0} y = \lim_{x\to0}\left(\frac{1}{x}\right)^{\tan x} \quad (\infty^0)$.

Taking ln on both sides, we get

$$\ln\left(\lim_{x\to0} y\right) = \lim_{x\to0}\left\{\tan x\ln\left(\frac{1}{x}\right)\right\} \qquad (0\times\infty)$$

$$= \lim_{x\to0}\frac{\ln\left(\frac{1}{x}\right)}{\cot x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to0}\frac{x\left(-\frac{1}{x^2}\right)}{-\csc^2 x} = \lim_{x\to0}\frac{1/x}{\csc^2 x} = \lim_{x\to0}\frac{\sin^2 x}{x} = \lim_{x\to0}(x)\left(\frac{\sin x}{x}\right)^2 = (0)(1) = 0$$

Thus, $\lim_{x\to0}\left(\frac{1}{x}\right)^{\tan x} = e^0 = 1$. [This is done by taking antilog on both sides]

5.4 APPLICATIONS OF LIMITS (ASYMPTOTES)

In this section we shall show a very useful application of the limits known as "Asymptotes". It may be noted that asymptotes help us to draw the graph of a function without tabulating its values. The asymptotic behavior of any curve is very much important in engineering and natural sciences. Asymptotes can be defined formally using the idea of limit in CALCULUS as:

Definition: A straight line L is called an asymptote for a curve C if the distance between L and C approaches zero as the distance moved along L from some fixed point on L tends to infinity. There are three types of asymptotes (i) Horizontal asymptote (ii) Vertical asymptote and (iii) Oblique asymptote. We now discuss how to find these asymptotes.

i. Horizontal Asymptote

A line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$

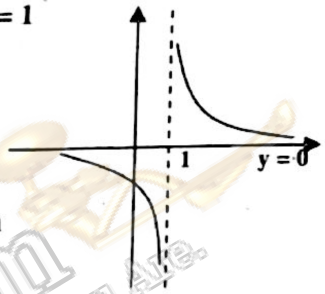
ii. Vertical Asymptote

A line $x = a$ is a vertical asymptote of the curve $y = f(x)$ if

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty.$$

REMARK: Horizontal asymptote is parallel to the x–axis and vertical asymptote is parallel to the y–axis.

Example 01: Find the horizontal and vertical asymptotes of the curves defined by the following equations:



Solution: (i) f(x) = 1/(x – 1)

Horizontal asymptote

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x-1} = 0 \text{ or } \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x-1} = 0.$$

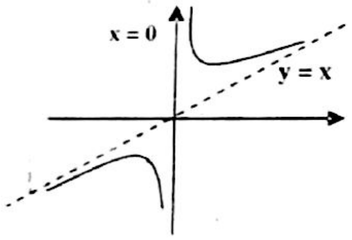
It implies that y = 0 (x-axis) is a horizontal asymptote for the given function.

Vertical asymptote

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = +\infty \text{ or } \mathrm{lLim}_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

It implies that $x = 1$ is a vertical asymptote for the given f.

(ii) f(x) = (x + 1/x)



Horizontal asymptote

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(x + \frac{1}{x} \right) = \infty$$

Thus there is no horizontal asymptote for the given curve.

Vertical asymptote

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(x + \frac{1}{x} \right) = \infty$$

Thus x = 0 (y–axis) is a vertical asymptote for the given curve.

REMARK: The graph of this function is shown above. This function has two asymptotes: the vertical asymptote and the oblique asymptote. We shall now discuss the method to find an oblique asymptote.

Oblique asymptote

An asymptote that is neither parallel to x–axis nor parallel to y–axis is said to be an oblique asymptote. The equation of such asymptote is of the form y = m x + c.

It may be noted that the graph of a rational function has an oblique asymptote if numerator is of one degree greater than that of the denominator. Consider a rational function defined by the equation

$$f(x) = \frac{x^2 + 1}{x} \qquad (1)$$

Here degree of the numerator is one greater than the degree of the denominator.

Let $y = mx + c$ be an oblique asymptote of the curve (1). Then as the curve (1) and the line $y = mx + c$ coincide while $x \to \infty$, we have

$mx+c=\frac{x^2+1}{x}\Rightarrow mx^2+cx=x^2+1\Rightarrow(m-1)x^2+cx-1=0$ (2)

Dividing by x^2, we get. $(m-1)+\frac{c}{x}-\frac{1}{x^2}=0$

Since $x\to\infty$, so we have $m-1=0$ or $m=1$. Put this value of m in equation (2) we get $cx-1=0\Rightarrow c=1/x$. Since $x\to\infty$ we get $c=0$. Substituting these values of m and c in $y=mx+c$, we get $y=x$, which is the required oblique asymptote of the curve (1).

Note: $f(x)=\frac{x^2+1}{x}\Rightarrow f(x)=x+\frac{1}{x}$, which is the same curve as given in the above example where it was mentioned that $f(x)=x$ is an oblique asymptote of the given curve.

Another Method to Find an Oblique Asymptote

If the degree of the numerator of a rational function f(x)/g(x) is one more than the degree of the denominator, we can write $\frac{f(x)}{g(x)}=\varphi(x)+\frac{h(x)}{g(x)}$.

$\varphi(x)$ is the oblique asymptote of the graph of the function $f(x)/g(x)$.

Example 02: Find the oblique asymptote of $f(x)=\frac{x^2-4}{x-1}$

Solution: $f(x)=\frac{x^2-4}{x-1}\Rightarrow f(x)=(x+1)+\frac{-3}{x-1}$.

[Students are advised to divide (x^2-4) by $(x-1)$ and see the result]. It implies that the oblique asymptote for the given curve is $\phi(x)=x+1$.

Example 03: Find all possible asymptotes to the curve $y=\frac{x^2+3}{x}$

Solution: Horizontal asymptote

$\lim_{x\to\infty}y=\lim_{x\to\infty}\left(\frac{x^2+3}{x}\right)=\lim_{x\to\infty}\frac{x(x+3/x)}{x}=\infty\Rightarrow$ there is no horizontal asymptote for the given function.

Vertical asymptote

$\lim_{x\to0}y=\lim_{x\to0}\left(\frac{x^2+3}{x}\right)=\infty\Rightarrow x=0$ is a vertical asymptote for the given curve.

Oblique asymptote

$y=\frac{x^2+3}{x}\Rightarrow y=x+\frac{3}{x}\Rightarrow y=x$ is an oblique asymptote for the given function.

Note: When the equation of the curve is given in the implicit form f(x, y) = 0, then there is another method to find the horizontal and vertical asymptote for a given function that is explained in the following example.

Example 04: Find the horizontal and vertical asymptotes for the following curves:

(i) $x^3-x^2y-x-6=0$

(ii) $(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0$.

Solution: (i) Horizontal asymptote

Here the term of the highest power of x is x^3, put its coefficient equal to zero, that is; 1 = 0 which is not possible. It shows that the given curve does not have any horizontal asymptote.

Vertical asymptote

Here the term of the highest power of y is y, put its coefficient equal to zero, that is; $-x^2 = 0 \Rightarrow x = 0$.

Thus, $x = 0$ is the vertical asymptote for the given curve.

(ii) Simplifying the given equation

$$(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0$$

$$\Rightarrow (x^2-2xy+y^2)(x^2+y^2)-10x^3+10x^2y+12y^2+2x+y=0$$

$$\Rightarrow x^4-2x^3y+x^2y^2+x^2y^2-2xy^3+y^4-10x^3+10x^2y+12y^2+2x+y=0$$

or, $x^4+y^4-(2y+10)x^3+(2y^2+10y)x^2-(2y^3-2)x+12y^2+y=0$ (1)

Horizontal asymptote

Here the term of the highest power of x is x^4, put its coefficient equal to zero, that is; 1= 0 which is not possible. It shows that the given curve does not have any horizontal asymptote.

Vertical asymptote

Here the term of the highest power of y is y^4, put its coefficient equal to zero, that is; 1 = 0 which is not possible. It shows that the given curve does not have any vertical asymptote.

WORKSHEET 05

1. Verify the Rolle's Theorem and find c (if possible) for the following functions where interval for each function is also given.

(i) $f(x)=x^2-3x+2,\ [1,2]$ (ii) $f(x)=\sin^2 x,\ [0,\pi]$ (iii) $f(x) = 1 - x^{3/4}$, [-1, 1]

(iv) $f(x)=\dfrac{1-x^2}{1+x^2}\ [-1,1]$ (v) $f(x)=x(x+3)e^{-\frac{x}{2}}\ [-3,0]$ (vi) $f(x)=2+(x-1)^{3/2}\ [0,2]$

(vii) $f(x)=x^2-6x+8\ [2,4]$ (viii) $f(x)=x^3-4x\ [-2,2]$ (ix) $f(x)=8x-x^2\ [0,8]$

(x) $f(x)=x^2\ [-1,1]$ (xi) $f(x)=\sin x\ [-\pi,\pi]$ (xii) $f(x)=e^x\ [0,\pi]$

(xiii) $f(x)=\dfrac{\sin x}{e^x}\ [0,\pi]$ (xiv) $f(x)=\sqrt{4-x^2}\ [-2,2]$ (xv) $f(x)=\tan x\ [0,\pi]$

(xvi) $f(x)=x^2-7x+12\ [3,4]$ (xvi) $f(x)=x(x-2)^3\ [0,2]$

(xvii) $f(x)=\sec x\ [0,2\pi]$ (xviii) $f(x)=x^3-3x^2+3x+2\ [1,2]$

2. Discuss the applicability of Roll's Theorem to the function:

$F(x) = |x|$, [-1, 1]

3. Find c (if possible) of the Mean Value Theorem for the following functions:

(i) $f(x)=x^3-3x^2+3x+2\ [1,2]$ (ii) $f(x)=x^2-7x+12\ [3,4]$ (iii) $f(x)=x(x-2)^3\ [0,2]$

(iv) $f(x)=x^3-3x-1\ \left[-\frac{11}{7},\frac{13}{7}\right]$ (v) $f(x)=\sqrt{x-2}\ [2,4]$ (vi) $f(x)=x^3-5x^2+4x-2\ [1,3]$

(vii) $f(x)=x^{2/3}$ $[-1,1]$ (viii) $f(x)=x(x-1)(x-2)$ $\left[0,\frac{1}{2}\right]$ (ix) $f(x)=(x-1)(x-2)(x-3)$ $[0,4]$

(x) $f(x)=\sqrt{x^2-4}$ $[2,4]$ (xi) $f(x)=\log x$ $[1,e]$ (xii) $f(x)=e^x$ $[0,1]$

(xiii) $f(x)=x^3-5x^2-3x$ $[1,3]$ (xiv) $f(x)=\sqrt{x^2-9}$ $[3,4]$

4. Find the Maclaurin series of the following functions:

(i) $f(x)=\sin x$ (ii) $f(x)=\cos x$ (iii) $f(x)=\sec x$

(iv) $f(x)=\tan x$ (v) $f(x)=\ln(1-x)$ (vi) $f(x)=e^{\sin x}$

5. Apply Taylor's Theorem to prove that:

(i) $a^{x+h}=a^x\left[1+h\log a+\frac{h^2}{2!}(\log a)^2+\frac{h^3}{3!}(\log a)^3+\cdots\right]$

(ii) $e^{x+h}=e^x\left[1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots\right]$ (iii) $\frac{1}{x+h}=\frac{1}{x}\left[1-\frac{h}{x}+\frac{h^2}{x^2}-\frac{h^3}{x^3}+\cdots\right]$

(iv) $\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\cdots$

(v) $\ln\sin(x+h)=\ln\sin x+h\cot x-\frac{h^2}{2}\csc^2 x+\frac{h^3}{3}\cot x\csc^2 x+\cdots$

(vi) $\ln\cos(x+h)=\ln\cos x-h\tan x-\frac{h^2}{2}\sec^2 x-\frac{h^3}{3}\sec^2 x\tan x+\cdots$

(vii) $\sin^{-1}(x+h)=\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+\frac{x}{(1-x^2)^{3/2}}\cdot\frac{h^2}{2!}+\cdots$

(viii) $\tan^{-1}(x+h)=\tan^{-1}x+\frac{h}{1+x^2}-\frac{xh^2}{(1+x^2)^2}+\cdots$

(ix) $\sec^{-1}(x+h)=\sec^{-1}x+\frac{h}{x\sqrt{x^2-1}}-\frac{h^2}{2!}\cdot\frac{2x^2-1}{x^2(x^2-1)^{3/2}}+\cdots$

(x) $\tan(x+h)=\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}\sec^2 x(1+3\tan^2 x)+\cdots$

6. Prove the following:

(a) $\lim_{x\to4}\frac{x^4-256}{x-4}=256$ (b) $\lim_{x\to4}\frac{x^4-256}{x^2-16}=32$ (c) $\lim_{x\to3}\frac{x^2-3x}{x^2-9}=\frac{1}{2}$

(d) $\lim_{x\to2}\frac{e^x-e^2}{x-2}=e^2$ (e) $\lim_{x\to0}\frac{xe^x}{1-e^x}=-1$ (f) $\lim_{x\to0}\frac{e^x-1}{\tan 2x}=\frac{1}{2}$ (g) $\lim_{x\to-1}\frac{\ln(2+x)}{x+1}=1$

(h) $\lim_{x\to0}\frac{\cos x-1}{\cos 2x-1}=\frac{1}{4}$ (i) $\lim_{x\to0}\frac{e^{2x}-e^{-2x}}{\sin x}=4$ (j) $\lim_{x\to0}\frac{8^x-2^x}{4x}=\frac{1}{2}\ln 2$

(k) $\lim_{x\to0}\frac{2\tan^{-1}x-x}{2x-\sin^{-1}x}=1$ (l) $\lim_{x\to0}\frac{\ln\sec 2x}{\ln\sec x}=4$

(m) $\lim_{x\to0}\frac{\ln\cos x}{x^2}=-\frac{1}{2}$ (n) $\lim_{x\to0}\frac{\cos 2x-\cos x}{\sin^2 x}=-\frac{3}{2}$ (o) $\lim_{x\to+\infty}\frac{\ln x}{\sqrt{x}}=0$

(p) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\csc 6x}{\csc 2x} = \dfrac{1}{3}$ (q) $\lim\limits_{x\to+\infty} \dfrac{5x+2\ln x}{x+3\ln x} = 5$ (r) $\lim\limits_{x\to+\infty} \dfrac{x^4+x^2}{e^x+1} = 0$

(s) $\lim\limits_{x\to0^+} \dfrac{\ln \cot x}{e^{\csc^2 x}} = 0$ (t) $\lim\limits_{x\to+\infty} \dfrac{e^x+3x^3}{4e^x+2x^2} = \dfrac{1}{4}$ (u) $\lim\limits_{x\to0} (e^x-1)\cos x = 1$

(v) $\lim\limits_{x\to-\infty} x^2 e^x = 0$ (w) $\lim\limits_{x\to0} x\csc x = 1$ (x) $\lim\limits_{x\to1} \csc \pi x \ln x = -\dfrac{1}{\pi}$

(y) $\lim\limits_{x\to\frac{\pi}{2}} e^{-\tan x}\sec^2 x = 0$ (z) $\lim\limits_{x\to0}\left(x-\sin^{-1}x\right)\csc^3 x = -\dfrac{1}{6}$ (a′) $\lim\limits_{x\to2}\left(\dfrac{4}{x^2-4}-\dfrac{1}{x-2}\right) = -\dfrac{1}{4}$

(b′) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right) = 0$ (c′) $\lim\limits_{x\to\frac{\pi}{2}}\left(\sec^3-\tan^3 x\right) = \infty$ (d′) $\lim\limits_{x\to1}\left(\dfrac{1}{\ln x}-\dfrac{x}{x-1}\right) = -\dfrac{1}{2}$

(e′) $\lim\limits_{x\to0}\left(\dfrac{4}{x^2}-\dfrac{2}{1-\cos x}\right) = -\dfrac{1}{3}$ (f′) $\lim\limits_{x\to+\infty}\left(\dfrac{\ln x}{x}-\dfrac{1}{\sqrt{x}}\right) = 0$ (g′) $\lim\limits_{x\to0^+} x^x = 1$

(h′) $\lim\limits_{x\to0}(\cos x)^{1/x} = 1$ (i′) $\lim\limits_{x\to0}\left(e^x+3x\right)^{1/x} = e^4$ (j′) $\lim\limits_{x\to+\infty}\left(1-e^{-x}\right)^{e^x} = \dfrac{1}{e}$

(k′) $\lim\limits_{x\to x\to\pi/2} (\sin x-\cos x)^{\tan x} = \dfrac{1}{e}$ (l′) $\lim\limits_{x\to\pi/2} (\tan x)^{\cos x} = 1$ (m′) $\lim\limits_{x\to1} x^{\tan\frac{\pi x}{2}} = e^{-2/\pi}$

(n′) $\lim\limits_{x\to+\infty}(1+1/x)^x = e$ (o′) $\lim\limits_{x\to+\infty} \dfrac{2^x}{3^x} = 0$ (p′) $\lim\limits_{x\to0^+} \dfrac{e^{-3/x}}{x^2} = 0$

(q′) $\lim\limits_{x\to0} \dfrac{e^x\left(1-e^x\right)}{(1+x)(1-x)} = \lim\limits_{x\to0} \dfrac{e^x}{(1+x)} \lim\limits_{x\to0} \dfrac{\left(1-e^x\right)}{(1-x)} = 1$

7. Determine the horizontal and vertical asymptotes for the following functions:

(i) $y = \dfrac{(x-2)^2}{x^2}$ (ii) $x^2y^2 = 12(x-3)$ (iii) $2xy = x^2+3$

(iv) $x^2(x-y)^2 + a^2\left(x^2-y^2\right) = a^2xy$

(v) $(x-y)^2\left(x^2+y^2\right) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$

(vi) $x^2y + xy^2 + xy + y^2 + 3x = 0$

8. Find all possible asymptotes of $2xy + 2y = (x-2)^2$

CHAPTER SIX

FURTHER APPLICATIONS OF DERIVATIVES

In this chapter, we shall present some important applications of derivatives that help to understand how the derivatives are used in solving engineering and technical problems.

6.1 TANGENT AND NORMAL

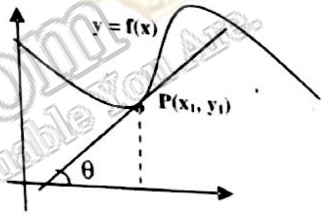
Definition: A straight line that touches the curve $y = f(x)$ at only one point is called tangent to the curve $y = f(x)$ at that point. The following figure shows the tangent line at point P. Equation of the tangent line can be derived as shown below:

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$.

We assume that f is derivable at P so that $f'(x)$ exists.

At $P(x_1, y_1)$ the slope of the tangent is the value of dy/dx at this point. Let this value be denoted by m.

Now the tangent at P is a line through $P(x_1, y_1)$ having slope m is given by:

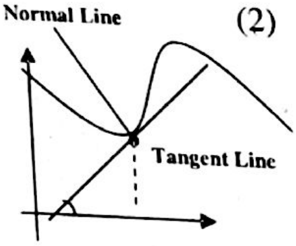


$$y - y_1 = m(x - x_1) \quad \text{[using point-slope form]} \qquad (1)$$

where $m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$

This is the equation of tangent line to the curve $y = f(x)$ at (x_1, y_1).

Definition: The normal to a curve at any point $P(x_1, y_1)$ is the straight line through the point $P(x_1, y_1)$ perpendicular to the tangent to the curve at that point. Equation of the normal is: $y - y_1 = -\frac{1}{m}(x - x_1)$ [using point-slope form] (2)



Here m is given as above.

Example 01: Find the equation of tangent and normal to the curve xy = 10 at (1, 10).

Solution: Equation of curve is: $xy = 10$ (1)

and $(x_1, y_1) = (1, 10)$.

Differentiating (1) with respect to x, we get

$$xy' + y = 0 \Rightarrow xy' = -y \Rightarrow y' = -y/x$$

At (1, 10), $m = dy/dx = -10/1 = -10$.

Equation of tangent: $y - y_1 = m(x - x_1)$

Substituting the values, we get: $y - 10 = -10(x - 1) \Rightarrow 10x + y - 20 = 0$

This is the equation of tangent line for the given curve.

Equation of normal: $y - y_1 = -\frac{1}{m}(x - x_1)$

Substituting the values, we get: $y - 10 = -\frac{1}{-10}(x - 1) \Rightarrow x - 10y + 99 = 0$

This is the equation of normal line for the given curve.

Example 02: Find the equations of tangent and normal to the curve $x^3 + xy^2 - ay^2 = 0$ at $x = a/2$.

Solution: Putting $x = a/2$ in the given equation, we get

$$(a/2)^3 + (a/2)\,y^2 - ay^2 = 0 \Rightarrow -(a/2)\,y^2 = -(a/2)^3 \Rightarrow y^2 = (a/2)^2 \Rightarrow y = \pm\, a/2.$$

Hence, points where the tangent and normal are to be found are $(a/2, a/2)$ and $(a/2, -a/2)$. Now from the given equation, we have $f(x, y) = x^3 + xy^2 - ay^2$.

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 + y^2}{2xy - 2ay}$$

At $(a/2, a/2)$, $m = \dfrac{dy}{dx} = -\dfrac{3(a/2)^2 + (a/2)^2}{2(a/2)(a/2) - 2a(a/2)} = 2$.

Hence equation of tangent line is: $(y - a/2) = 2\,(x - a/2)$ $\Rightarrow 4x - 2y = a$.

The equation of normal is: $(y - a/2) = -1/2\,(x - a/2)$ $\Rightarrow 2x + 4y = 3a$

Similarly the equations of tangent and normal at $(a/2, -a/2)$ are:

$4x + 2y = 2$ and $2x - 4y = 3a$ respectively.

Example 03: Find the equations of tangent and normal to the curve $y = 3x^2 + 5x$ touching the y – axis.

Solution: Equation of curve is: $y = 3x^2 + 5x$ (1)

Since the curve touches the y-axis, therefore $x = 0$. Substituting $x = 0$ into (1), we get: $y = 3(0) + 5(0) = 0$. Thus the point where the tangent and normal are to be found is $(x_1, y_1) = (0, 0)$.. To find these equations we first find the slope of the curve at $(0, 0)$. Differentiating (1) with respect to x, we get: $dy/dx = 6x + 5$

At $(0, 0)$: $m = dy/dx = 6(0) + 5 = 5$

Equation of tangent: $y - y_1 = m(x - x_1)$

Substituting the values, we get: $y - 0 = 5(x - 0) \Rightarrow 5x - y = 0$

This is the equation of the tangent line for the given curve.

Equation of normal: $y - y_1 = -\dfrac{1}{m}(x - x_1)$

Substituting the values, we get: $y - 0 = -\dfrac{1}{5}(x - 0) \Rightarrow x + 5y = 0$

This is the equation of the normal line for the given curve.

Example 04: Find the points where the tangent is parallel to the x–axis and where it is parallel to the y–axis for the following curve $x^3 + y^3 = a^3$.

Solution: Equation of curve is $x^3 + y^3 = a^3$ (1)

Differentiating with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow x^2 + y^2\frac{dy}{dx} = 0 \Rightarrow y^2\frac{dy}{dx} = -x^2 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \qquad (2)$$

If the tangent is parallel to x–axis then $\dfrac{dy}{dx} = 0 \Rightarrow 0 = -\dfrac{x^2}{y^2} \Rightarrow x^2 = 0 \Rightarrow x = 0$.

Substituting $x = 0$ into (1), we get: $0 + y^3 = a^3 \Rightarrow y^3 = a^3 \Rightarrow y = a$.

This shows that the point where the tangent to the given curve is parallel to the x-axis is $(0, a)$. If the tangent is parallel to y–axis then its slope is infinity, that is;

$dy/dx = \infty \quad \Rightarrow dy/dx = -x^2/y^2 = \infty \quad \Rightarrow \quad y = 0$

Substituting y = 0 into (1), we get: $x^3 + 0 = a^3 \Rightarrow x^3 = a^3 \Rightarrow x = a$.
This shows that the point where the tangent to the given curve is parallel to the y-axis is (a, 0).

Example 05: Find the angle of intersection between the curves $y^2 = 4ax$ and $x^2 = 4ay$ at the point other than (0, 0).

Solution: It may be noted that angle between two curves is the angle between their tangent lines at the point of their intersection. Now the point of intersection of the given curves can be found by solving their equations simultaneously.

Given: $y^2 = 4ax$ (1)

and $x^2 = 4ay$ (2)

From (2), $y = x^2/4a$. Substituting this in (1) and simplify, we get x = 4a.
Putting this in (2) gives y = 4a. Hence the point where two curves intersect is (4a, 4a).
Now differentiating (1) w.r.t x, we get

$$dy/dx = 2a/y \quad (3)$$

Similarly, differentiating (2) w.r.t x, we get

$$dy/dx = x/2a \quad (4)$$

At (4a, 4a): Form (3) $m_1 = 1/2$ and from (4) $m_2 = 2$

If θ is the angle between two tangent lines, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{2 - 1/2}{1 + 2(1/2)} = \frac{3/2}{2} = \frac{3}{4} \Rightarrow \theta = \tan^{-1}(0/0.75) \approx 37^0$$

Example 06: Find the equation of tangent at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Also show that portion of this tangent line between the x and y axes is always constant.

Solution: Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. Differentiating w.r.t x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \quad \rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Thus equation of tangent is: $(Y - y) = -(y/x)^{1/3}(X - x)$ (1)

If the tangent line cuts the x-axis then Y = 0

➔ $X = x + x^{1/3}y^{2/3} = x^{1/3}(x^{2/3} + y^{2/3}) = x^{1/3}a^{2/3}$ (NOTE: $x^{2/3} + y^{2/3} = a^{2/3}$)

Thus x-intercept of tangent line is $P(x^{1/3}a^{2/3}, 0)$

If the tangent line cuts the y-axis then X = 0

➔ $Y = y + y^{1/3}x^{2/3} = y^{1/3}(y^{2/3} + x^{2/3}) = y^{1/3}a^{2/3}$ (NOTE: $x^{2/3} + y^{2/3} = a^{2/3}$)

Thus y-intercept of tangent line is $Q(0, y^{1/3}a^{2/3})$.

Thus portion of the tangent line between the x and y axes is |PQ| and is given by using the distance between two points:

$$|PQ| = \sqrt{\left(x^{1/3}a^{2/3} - 0\right)^2 + \left(0 - y^{1/3}a^{2/3}\right)^2} = \sqrt{a^{4/2}\left(x^{2/3} + y^{2/3}\right)} = \sqrt{a^{4/3}.a^{2/3}} = \sqrt{a^2} = a$$

Thus |PQ| = a (Constant).

Example 07: Find the equation of normal at any point θ to the curve

x = a(cos θ + θ sin θ), y = a(sin θ – θ cos θ)

Verify that these normals touch a circle with its center at the origin and whose radius is constant.

Solution: First of all let us see that given equation represents which curve. Squaring and adding both equations and simplifying, we get: $x^2 + y^2 = a^2(1 + \theta^2)$

This is equation of circle centered at origin (0, 0) with radius $r = a\sqrt{1+\theta^2}$. This means for different values of θ given equations represent family of circles.

For example, if θ → 0, we get: $x^2 + y^2 = a^2$

if $\theta = 1$, we get: $x^2 + y^2 = \sqrt{2}a^2$, etc.

Now given equations are the parametric equations of curve, hence

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} \quad (1)$$

$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$, $\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$

Thus $$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\sin\theta}{\cos\theta}$$

Now equation of normal is: $y - y_1 = -\frac{1}{m}(x - x_1)$. This gives,

$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}[x - a(\cos\theta + \theta\sin\theta)]$

➔ $y\sin\theta - a\sin^2\theta + a\theta\sin\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

➔ $x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta) = a$.

Thus equation of normal to given curve is: $x\cos\theta + y\sin\theta - a = 0$

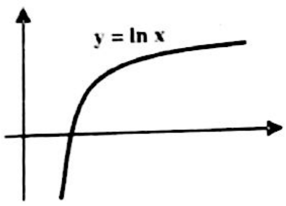
Now we know that distance between a straight line $ax + by + c = 0$ and a point (x_1, y_1) is:

$$D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Using this formula, we see that distance between normal $x\cos\theta + y\sin\theta - a = 0$ and the origin (0, 0) is: $$D = \frac{|0.\cos\theta + 0.\sin\theta - a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = a \text{ (constant)}$$

This shows that normal touches the circle of radius `a` with center at (0, 0).

Example 08: At what point is the tangent to the curve y = ln x parallel to the chord joining the points (0, 0) and (0, 1).



Solution: The graph of y = ln x is shown here.

Given that y = ln x ➔ $y' = 1/x = m_1$.

This is the slope of tangent line to the curve y = ln x.

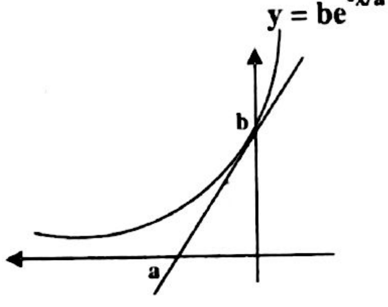
Now slope of the chord joining the points (0, 0) and (1, 0) is $m_2 = (y_2 - y_1)/(x_2 - x_1) = (1 - 0)/(0 - 0) = \infty$

Since tangent and chord are parallel hence their slopes are equal, that is; $m_1 = m_2$.

➔ $1/x = \infty$ ➔ $x = 0$. Put this in given equation, we get $y = \ln 0 = -\infty$. Thus required point is $(0, -\infty)$.

Example 09: Prove that x/a + y/b = 1 touches the curve $y = a e^{-x/a}$ at the point where the given curve crosses the y-axis.

Solution: Observe that x/a + y/b = 1 is an equation of line with x-intercept equal to `a` and y-intercept equal to `b`.



It is also given that the curve $y = b e^{-x/a}$ cuts the y-axis.

Now on y-axis, x is always zero. Thus putting x = 0, we get: $y = b e^0 = b$. Now the slope of given curve is:

$y' = -be^{-x/a}/a = -y/a = m_1$.

Given line may be expresses in slope intercept form as:

$y = (-b/a)x + b$. Thus slope of this line is $m_2 = -b/a$.

Since given line touches the curve hence $m_1 = m_2$.

➔ $-y/a = -b/a$ ➔ $y = b$, which is true.

Thus given line touches the given curve at the point where the curve crosses the y-axis.

Example 10: Show that the sum of the intercepts on the axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

Solution: Given curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Differentiate w.r.t x, we get:

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad y' = -\sqrt{\frac{y}{x}}.$$

Thus equation of tangent line to the given curve is: $(Y - y) = -\sqrt{y/x}\,(X - x)$

For x-intercept, put $Y = 0$ and simplify, we get: $X = \sqrt{xy} + x$

For y-intercept, put $X = 0$ and simplify, we get: $Y = \sqrt{xy} + y$

Now sum of the intercepts is:

$$X + Y = x + y + 2\sqrt{xy} = \left(\sqrt{x} + \sqrt{y}\right)^2 = \left(\sqrt{a}\right)^2 = a \text{ (constant)}$$

Example 11: If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_2, y_2), show that $(x_2/x_1) = (y_2/y_1)$

Solution: Given equation of curve is $x^3 + y^3 = a^3$. Differentiating w.r.t x, we get

$$3x^2 + 3y^2 y' = 0 \quad \Rightarrow \quad y' = -x^2/y^2$$

Then slope of tangent at (x_1, y_1) is: $m_1 = -(x_1)^2/(y_1)^2$

And slope of tangent at (x_2, y_2) is : $m_2 = -(x_2)^2/(y_2)^2$

Since the slope at (x_1, y_1) and (x_2, y_2) is same, hence $m_1 = m_2$

$\Rightarrow -(x_1)^2/(y_1)^2 = -(x_2)^2/(y_2)^2 \Rightarrow (x_1)/(y_1) = (x_2)/(y_2) \Rightarrow (x_2/x_1) = (y_2/y_1)$

Example 12: Find the equation of a tangent line to the curve $y = x^3 + 2x^2 + 1$ where it is parallel to the line $y = 1 - x$.

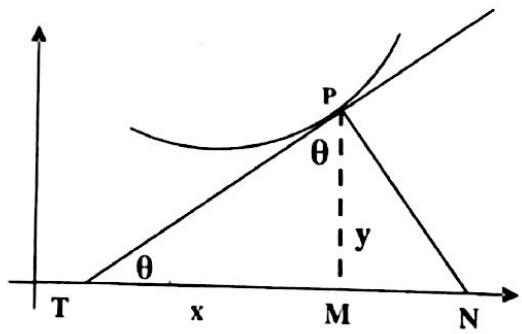
Solution: Differentiate given equation w.r.t x we get: $y' = 3x^2 + 4x$. This is the slope given curve. Let us call it m_1. The slope of line $y = 1 - x$ is $m_2 = -1$. Since the tangent and the line are parallel hence their slopes are equal, that is; $m_1 = m_2$. This implies that:

$$3x^2 + 4x = -1 \Rightarrow 3x^2 + 4x + 1 = 0 \Rightarrow x = -1 \text{ and } x = -1/3.$$

Put $x = -1$ in the given equation, we get: $y = 2$. Similarly put $x = -1/3$ and simplify, we get $y = 20/27$. Thus there are two point where the tangent to the given curve and the line $y = 1 - x$ are parallel and these points are: $(-1, 2)$ and $(-1/3, 20/27)$.

Lengths of Tangent, Normal, Sub-tangent and Sub-Normal

Consider a curve as shown in the adjacent figure. Let P be any point on this curve where the tangent is drawn which meets the x-axis at T. Draw a normal on the tangent at P and produce it to meet the x-axis at N. Also draw PM perpendicular on the x-axis.



Then TM is called sub-tangent and MN is called sub-normal.

Let angle MPT = θ then angle MPN = θ.

Also ΔTMP and ΔNMP are right triangles.

Before we find the lengths of tangent, normal, sub-tangent and sub-normal the following may be noted:

Slope of tangent line is $m = \tan\theta = dy/dx \rightarrow$ Slope of normal is $-\cot\theta = -dx/dy$

Also $\operatorname{cosec}^2\theta = 1 - \cot^2\theta = 1 - (dx/dy)^2$ and $\sec^2\theta = 1 + \tan^2\theta = 1 + (dy/dx)^2$

(1) From Δ TMP: $\sin\theta = |MP|/|PT| \Rightarrow |PT| = |MP|/\sin\theta = y\operatorname{cosec}\theta$

Thus length of tangent is: $|PT| = y\sqrt{1-\cot^2\theta} = y\sqrt{1-(dx/dy)^2}$

(2) From Δ NMP: sin θ = |MP|/|PN| ➔ |PN| = |MP|/cos θ = y sec θ

Thus length of normal is: $|PN| = y\sqrt{1-\tan^2\theta} = y\sqrt{1-(dy/dx)^2}$

(3) From Δ TMP: cot θ = |TM|/|MP| ➔ |TM| = |PM| cot θ = y cot θ
Thus length of sub-tangent is: |TM| = y(dx/dy)
(4) From Δ TMN: tan θ = |MN|/|MP| ➔ |MN| = |MP| tan θ = y tan θ
Thus length of sub-normal is: |MN| = y(dy/dx)

Example 12: For the curve x = a(cos θ + ln tan θ/2), y = a sin θ, prove that length of tangent is constant. Also find the length of sub-tangent, normal and sub-normal.

Solution: $\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\theta/2}.\sec^2\theta/2.\frac{1}{2}\right) = a\left(-\sin\theta + \frac{1}{2\sin\theta/2\cos\theta/2}\right)$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = a\left(\frac{-\sin^2\theta+1}{\sin\theta}\right) = a\left(\frac{\cos^2\theta}{\sin\theta}\right)$$

Also $\frac{dy}{d\theta} = a\cos\theta \Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = a\cos\theta \div a\left(\frac{\cos^2\theta}{\sin\theta}\right) = \frac{a\cos\theta.\sin\theta}{a\cos^2\theta} = \tan\theta$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = a\cos\theta \div a\frac{\cos^2\theta}{\sin\theta} = a\cos\theta \times \frac{\sin\theta}{a\cos^2\theta} = \tan\theta \text{ and } \frac{dx}{dy} = \cot\theta$$

Length of tangent = $y\sqrt{1-(dx/dy)^2} = a\sin\theta\sqrt{1-\cot^2\theta} = a\sin\theta.\operatorname{cosec}\theta = a$ (const)

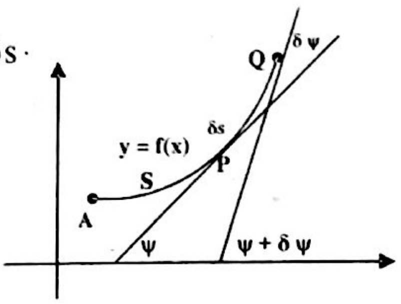
Length of sub-tangent = y.(dx/dy) = a sin θ.cot θ = a sin θ (cos θ/sin θ) = a cos θ

Length of normal is: $y\sqrt{1-(dy/dx)^2} = a\sin\theta\sqrt{1-\tan^2\theta} = a\sin\theta\sec\theta = a\tan\theta$

Length of sub-normal is = y(dy/dx) = a sin θ.tan θ

6.2 CURVATURE AND RADIUS OF CURVATURE

Let P and Q be two neighboring points on a curve.
Let arc AP = s, and arc $AQ = s + \delta s$ so that arc $PQ = \delta s$.
Let 'A' be a fixed point on the curve from where the arcs are measured. Let the tangents at P and Q make angles ψ and ψ + δ ψ respectively with x–axis.
In moving from P to Q through a distance δs, the tangent has turned through the angle δ ψ. This is called total bending or total curvature of the arc PQ.
Therefore, average curvature of the arc PQ = $\delta\psi/\delta s$.



The limiting value of average curvature when Q approaches P, is defined as the curvature of the curve at P.

Thus, the curvature κ (kappa) at the point P = $\lim_{Q\to P} \delta\psi/\delta s = \lim_{\delta s\to 0} \delta\psi/\delta s = d\psi/ds$.

Radius of curvature

The reciprocal of the curvature of the curve at P (provided this curvature is not zero) is called the Radius of the Curvature of the curve at P and is usually denoted by a Greek alphabet ρ (Rho). Thus,

$$\rho = \frac{1}{|\kappa|} = \left|\frac{ds}{d\psi}\right|.$$ [Note: Radius is always positive.]

REMARKS:

i. The curvature of a straight line is always zero. Hence the radius of the curvature of the straight line is infinite (not defined).

ii. The radius of curvature of the circle is simply its radius.

iii. Since $\delta\psi$ is measured in radians, the unit of curvature is radian per unit length, eg. radians per cm. Hence unit of radius of curvature is cm/radian.

We now write the formulae to compute the radius of curvature when the equations of the curves are given in different forms. The proofs may be found in any "Text Book" of Calculus.

(1) Explicit equation: If the equation of the curve is given in the explicit form $y = f(x)$, then

$$\rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right|.$$

(2) Implicit equation: If the equation of the curve is given in the implicit form, $f(x, y) = 0$, then

$$\rho = \left|\frac{\left[(f_x)^2+(f_y)^2\right]^{3/2}}{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}\right|$$

(3) Parametric equations: If the equation of the curve is given in parametric form, that is; $x = f(t)$, $y = g(t)$, then

$$\rho = \frac{\left[\{f'(t)\}^2+\{g'(t)\}^2\right]^{3/2}}{\left|f'(t)\cdot g''(t) - g'(t)\cdot f''(t)\right|}.$$

(4) Polar equation of a curve: If the equation of the curve is given in polar form $f(r, \theta)$, then

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\cdot\frac{d^2r}{d\theta^2}\right|}.$$

Example 01: Find the radius of curvature of the curve $f(x) = x^2 - 5x - 6$ at the point (4, -10).

Solution: The equation of the curve is given in explicit form, that is;

$$f(x) = x^2 - 5x - 6 \qquad (1)$$

Differentiating (1) with respect to x, we get: $f'(x) = 2x - 5$. At (4, -10), $f'(x) = 3$.

Differentiating with respect to x again, we get: $f''(x) = 2$.

$$\text{Now, } \rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right| = \left|\frac{\left[1+(3)^2\right]^{3/2}}{2}\right| = \left|\frac{(10)^{3/2}}{2}\right| \approx 15.8.$$

Hence, the radius of curvature of given curve at the given point is 15.8 cm/radian approximately.

Example 02: Find the radius of curvature of the curve y = 2x³ – 6x² + 11x at the point (1, 7).

Solution: Equation of the curve is given in explicit form, that is;

$$f(x) = 2x^3 - 6x^2 + 11x \qquad (1)$$

Differentiating (1) with respect to x, we get

$$f'(x) = 6x^2 - 12x + 11. \text{ At } (1, 7),\ f'(x) = 6(1)^2 - 12(1) + 11 = 5$$

Differentiating again with respect to x, we get

$$f''(x) = 12x - 12. \text{At } (1, 7),\ f''(x) = 12(1) - 12 = 0.$$

Now radius of curvature in explicit form is: $\rho = \left| \frac{\left[1 + \{f'(x)\}^2\right]^{3/2}}{f''(x)} \right|$

Since f ``(x) = 0, hence ρ is undefined. It implies that radius of curvature of the given curve at the indicated point does not exist. However, the curvature of given curve is zero radian/cm.

Example 03: Find the radius of curvature at any point of the curve
x = a cos t, y = a sin t.

Solution: Equation of the curve is given in parametric form. Now
Differentiating w.r.t t, we get:

$$f'(t) = -a\sin t,\ g'(t) = a\cos t,\ f''(t) = -a\cos t,\ g''(t) = -a\sin t$$

Now, $\rho = \frac{\left[\{f'(t)\}^2 + \{g'(t)\}^2\right]^{3/2}}{|f'(t)\cdot g''(t) - g'(t)\cdot f''(t)|}$

Substituting the values, we get

$$\rho = \frac{\left[(-a\sin t)^2 + (a\cos t)^2\right]^{3/2}}{|(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)|} = \frac{\left[a^2\sin^2 t + a^2\cos^2 t\right]^{3/2}}{|a^2\sin^2 t + a^2\cos^2 t|} = \frac{(a^2)^{3/2}}{a^2} = a$$

NOTE: [$\sin^2 t + \cos^2 t = 1$]

Example 04: For the cycloid x = a(t + sin t), y = a(1 – os t) prove that the radius of curvature is given by $\rho = 4a\cos(t/2)$.

Solution: The given equations $x = f(t) = a(1 + \sin t)$ and $y = g(t) = a(1 - \cos t)$ are parametric equations. Hence differentiating twice with respect to t, we get

$$f'(t) = a(1 + \cos t), f''(t) = -a\sin t, g'(t) = a\sin t, g''(t) = a\cos t.$$

Now, $\rho = \frac{\left[\{f'(t)\}^2 + \{g'(t)\}^2\right]^{3/2}}{|f'(t)\cdot g''(t) - g'(t)\cdot f''(t)|}$.

Substituting the values, we get

$$\rho = \frac{\left[a^2(1 + \cos t)^2 + a^2\sin^2 t\right]^{3/2}}{|a(1 + \cos t)(a\cos t) - a\sin t(-a\sin t)|} = \frac{\left[a^2\left(1 + 2\cos t + \cos^2 t + \sin^2 t\right)\right]^{3/2}}{|a^2\cos t + a^2\cos^2 t + a^2\sin^2 t|}$$

$$\rho = \frac{a^3(2+2\cos t)^{3/2}}{|a^2(1+\cos t)|} = \frac{2^{3/2}a^3(1+\cos t)^{3/2}}{a^2(1+\cos t)} = 2\sqrt{2}a\sqrt{1+\cos t} = 2\sqrt{2}a\left(\sqrt{2}\cos t/2\right) = 4a\cos(t/2).$$

Example 05: Find the radius of curvature of the curve r = a/(1 + cos θ) at the point (a, π/2).

Solution: The equation of the curve is given in polar form, that is; $r = \dfrac{a}{1+\cos\theta}$ (1)

Differentiating (1) with respect to θ, we get

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{a}{1+\cos\theta}\right) = a\frac{d}{d\theta}(1+\cos\theta)^{-1} = \frac{a\sin\theta}{(1+\cos\theta)^2} \quad (2)$$

At $\left(a, \dfrac{\pi}{2}\right)$, $\left.\dfrac{dr}{d\theta}\right|_{\left(a,\frac{\pi}{2}\right)} = \dfrac{a\sin(\pi/2)}{(1+\cos\pi/2)^2} = a$, $\left[\text{Note: } \sin\dfrac{\pi}{2} = 1, \cos\dfrac{\pi}{2} = 0\right]$

Differentiating (2) with respect to θ, we get

$$\frac{d^2r}{d\theta^2} = a\frac{d}{d\theta}\left(\frac{\sin\theta}{(1+\cos\theta)^2}\right) = a\left[\frac{(1+\cos\theta)^2(\cos\theta) - \sin\theta(2)(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}\right]$$

$$\frac{d^2r}{d\theta^2} = \frac{a\cos\theta(1+\cos\theta) + 2a\sin^2\theta}{(1+\cos\theta)^3}$$

At $\left(a, \dfrac{\pi}{2}\right)$, $\left.\dfrac{d^2r}{d\theta^2}\right|_{\left(a,\frac{\pi}{2}\right)} = \dfrac{a\cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right) + 2a\sin^2\frac{\pi}{2}}{\left(1+\cos\frac{\pi}{2}\right)^3} = 2a$.

Now, $\rho = \dfrac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\cdot\dfrac{d^2r}{d\theta^2}\right|}$. Substituting the values, we get

$$\rho = \frac{\left[(a)^2 + (a)^2\right]^{3/2}}{\left|(a)^2 + 2(a)^2 - a(2a)\right|} = \frac{(2a^2)^{3/2}}{|a^2|} = \frac{2\sqrt{2}a^3}{a^2} = 2\sqrt{2}a \text{ cm/radians.}$$

Example 06: Prove that the radius of curvature at the point (2a, 2a) on the curve $x^2y = a(x^2 + y^2)$ is 2a.

Solution: Given function is given in implicit form. Here f(x, y) = = $ax^2 + a\,y^2 - x^2y$.

Thus: $f_x = 2ax - 2xy$, $f_y = 2ay - x^2$, $f_{xx} = 2a - 2y$,
$f_{yy} = 2a$, $f_{xy} = -2x$.

At (2a, 2a): $f_x = 2a(2a) - 2(2a)(2a) = -4a^2$, $f_y = 2a(2a) - (2a)^2 = 0$
$f_{xx} = 2a - 2(2a) = -2a$, $f_{yy} = 2a$, $f_{xy} = -2(2a) = -4a$.

$$\rho = \left|\frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2}\right| = \left|\frac{\left[(-4a^2)^2 + (0)^2\right]^{3/2}}{-2a(0)^2 - 2(-4a^2)(0)(-4a) + (2a)(-4a^2)^2}\right| = \frac{64a^6}{32a^5} = 2a$$

Circle of Curvature

Let $y = f(x)$ have a continuous derivative in a neighborhood of $x = a$ and a non-zero second derivative $f''(x)$ at $x = a$. Then both the curvature k and the radius of curvature ρ at the point (a, f(a)) of curve C defined by $y = f(x)$ exit and are non-zero. If $f''(x) > 0$, C is concave up and if $f''(a) < 0$ then the curve C is concave down at (a, f(a)). The center of curvature of C at (a, f(a)) is the point Q on the normal to C at (a, f(a)) on the concave side of C whose distance from (a, f(a)) is ρ. The circle of curvature of C at (a, f(a)) is the circle with center Q and radius ρ. This circle is also called the **Osculating circle** of the curve C at (a, f(a)). The osculating circle has the same tangent as the curve at (a, f(a)).

Concave up Concave down

Definition: Let $y = f(x)$ be twice derivable function with nonzero curvature at a point P(x, y). Then the coordinates (h, k) of the center of curvature at point P are given by:

$$h = x - y'\frac{1+(y')^2}{y''}, \qquad k = y + \frac{1+(y')^2}{y''}$$

Example 07: Find the center of the curvature of the curve defined by $y = x^3$ at the point (1, 1). Also find radius of curvature and equation of Osculating circle.

Solution: Since $y = x^3 \rightarrow y' = 3x^2$ and $y'' = 6x$.

At (1, 1), $y' = 3$ and $y'' = 6$. Thus coordinates of the center of curvature are:

$$h = x - y'\frac{1+(y')^2}{y''} = 1 - 3\frac{1+9}{6} = -4, \quad k = y\frac{1+(y')^2}{y''} = 1 + \frac{1+9}{6} = \frac{8}{3}$$

Thus center of the curvature is: (-4, 8/3).

$$\text{Radius of curvature is } \rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right| = \left[\frac{1+(3)^2}{6}\right]^{3/2} = \frac{5\sqrt{10}}{3}$$

The equation of Osculating circle is: $(x-h)^2 + (y-k)^2 = \rho^2$

$$\rightarrow \quad (x+4)^2 + \left(y - \frac{8}{3}\right)^2 = \left(\frac{5\sqrt{10}}{3}\right)^2 \Rightarrow 3x^2 + 3y^2 + 24x - 16y - 14 = 0$$

6.3 MAXIMA AND MINIMA of A FUNCTION of ONE VARIABLE

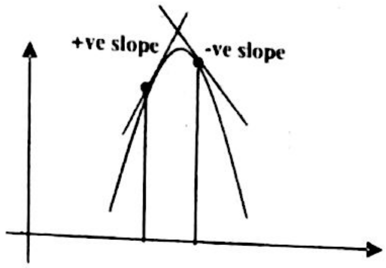
Applications of calculus to business, science, and industry are widespread. Our examples and exercises have been chosen to provide a feeling for how the derivatives can be used to solve real problems when the situation can be represented by a function.

Given a few guidelines, you will be able to look at the graph of a function and see where the function is increasing, where it is decreasing and, where it is maximum or minimum. For example, given a profit function curve, we have to observe when the profit is maximum/minimum or the profit is increasing or decreasing. Similarly, if we are given temperature curve against the time, you will be able to look at the graph and see when a metal is being heated or cooled. Learning how to interpret graphs will be an important experience, one filled with practical applications. A study of how derivatives apply to graphs will also enable to make many of the same determinations without the use of a graph.

Increasing and Decreasing Functions

A function is said to be increasing when its graph rises as it goes from left to right. A function is decreasing when

its graph falls as it goes from left to right.
The increasing or decreasing concept can be associated with the slope of the tangent line.
The slope of the tangent line to a curve will be positive when the curve is rising and negative when it is falling. This is shown in the figure.



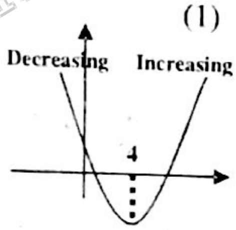
Since $f'(x)$ is the slope of the tangent line, it follows that if $f'(x) > 0$, then function f is increasing, and if $f'(x) < 0$, then f is decreasing.

Increasing/ Decreasing

1. At a point x = a where f(x) is defined (not infinite)
   (a) If $f'(a) > 0$, then f is increasing at $x = a$.
   (b) If $f'(a) < 0$, then f is decreasing at $x = a$.
2. On an interval where f is defined
   (a) If $f'(x) > 0$, for all x in an interval, then f is increasing on the interval.
   (b) If $f'(x) < 0$, for all x in an interval, then f is decreasing on the interval.

Example 01: Consider the function defined by $f(x) = x^2 - 8x + 7$, find the interval on which the function is increasing and the interval on which it is decreasing.

Solution: Given, $f(x) = x^2 - 8x + 7$ (1)



Differentiating (1) with respect to x, we get: $f'(x) = 2x - 8$
For f (x) to be increasing: $2x - 8 > 0 \Rightarrow 2x > 8 \Rightarrow x > 4$
Thus f(x) is increasing in the interval $(4, \infty)$.
For f(x) to be decreasing: $2x - 8 < 0 \Rightarrow 2x < 8 \Rightarrow x < 4$.
Thus f(x) is decreasing in the interval $(-\infty, 4)$.

Example 02: Let $T(x) = -3x^2 + 60x + 70$ be the temperature after x seconds of a metal tray undergoing a chemical finishing process. Determine when the metal being cooled and heated.

Solution: The metal is being cooled when the temperature T(x) is decreasing. We know that a function is decreasing when its derivative is less than zero, that is. Thus T(x) is decreasing when $T'(x) < 0$.

Now, $T(x) = -3x^2 + 60x + 70$, ➔ $T'(x) = -6x + 60$.
Hence, $-6x + 60 < 0 \Rightarrow -6x < -60 \Rightarrow x > 10$.
Thus, the metal is being cooled after 10 seconds.
You may observe that the temperature during first 10 seconds of metal tray is increasing, that is; $T'(x) > 0$ for $0 < x < 10$ and then it starts cooling after 10 seconds.

Example 03: Suppose that $P(x) = -0.01x^2 + 60x - 500$ is the profit from the manufacture and sale of x telephones. Is the profit increasing or decreasing when 100 phones have been sold?

Solution: Since, $P(x) = -0.01x^2 + 60x - 500$ (1)
Differentiating (1) with respect to x, we get: $P'(x) = -0.02x + 60$
When x = 100, we have: $P'(100) = -0.02(100) + 60 = 58$
Since, $P'(100) > 0$, this means that profit is increasing when 100 phone sets have been sold.

Relative Maxima and Minima

The absolute maximum value of a function is the largest possible value of the function. The absolute maximum value of a particular function may or may not be the same as the relative maximum value. Consider the function graphed in the figure 1 below, defined for x in the interval [2, 8]. The relative maximum value of the function is 5, but the absolute maximum value of the function is 9.

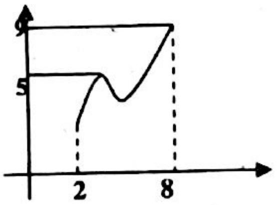


Fig. 1

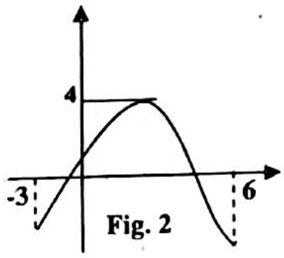


Fig. 2

Consider next the function graphed in figure 2. , defined for x in the interval [-3, 6].The relative maximum value of the function is 4, and this is also the absolute maximum value of the function.

i. The absolute maximum function value occurs either where there is a relative maximum or at an endpoint of the interval.

ii. The absolute minimum function value occurs either where there is a relative minimum or at an endpoint of the interval.

Similar drawings and reasoning can be used to present the absolute minimum versus relative minimum.

Maximum values of y = f(x)

A function f(x) is said to have a relative maximum value f(a) at x = a if f(x) increases before x = a and decreases after x = a.

Minimum values of y = f(x)

A function f(x) is said to have a relative minimum value f(a) at x = a if f(x) decreases before x = a and increases after x = a.

REMARKS:

(i) Maximum and minimum values are also called **extreme values or turning values or stationary values**.

(ii) The points where a function has a maximum or minimum value are called **turning points or stationary points.**

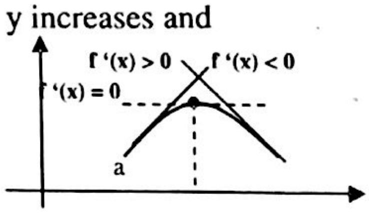
(iii) The values at which $f'(x)=0$, are called **stationary values or critical values.**

(iv) A point that is neither maximum nor minimum is called **point of inflexion or saddle point**.

Conditions for maximum and minimum values of a function:

First Derivative Test

(a) f(x) has a maximum value at x = a if f(x) increases before x = a and decreases as x increases beyond a. Thus, when x is slightly less than a, y increases and therefore $f'(x)$ is positive. When x is slightly greater than a, y decreases and therefore $f'(x)$ is negative. Therefore $f'(x)$ changes sign from positive to negative as x passes through the value a.

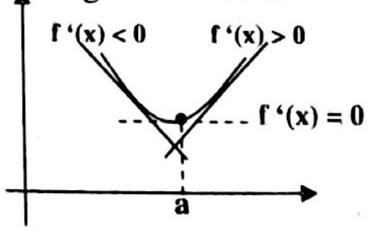


Hence we have the following two conditions for y = f(x) to have a maximum value f(a) at x = a.

(i) $f'(x) = 0$ at x = a.

(ii) $f'(x)$ changes sign from positive to negative as x passes through the value a.

(b) f(x) has a minimum value at x = a if f(x) decreases before x = a and increases as x increases beyond a. Thus, when x is slightly less than a, y decreases and therefore $f'(x)$ is negative. When x is slightly greater than a, y increases and therefore



$f'(x)$ is positive. Therefore $f'(x)$ changes sign from negative to positive as x passes through the value a. Hence we have the following two conditions for $y = f(x)$ to have a minimum value f(a) at x = a.

(i) $f'(x) = 0$ at x = a

(ii) $f'(x)$ changes sign from negative to positive as x passes through the value a.

Second Derivative Test

(a) As observed in the first derivative test, f(x) has maximum value at x = a if $f'(x)$ changes sign from positive to negative at x = a. But $f'(x)$ is itself a function of x which changes sign from positive to negative, therefore, it decreases at x = a and hence its derivative $f''(x)$ is negative at x = a. Hence a function y = f(x) has a maximum value at x = a if

(i) $f'(x) = 0$ at x = a

(ii) $f''(x)$ is negative at x = a

(b) As has been seen in the first derivative test, f(x) has a minimum value at x = a if $f'(x)$ changes sign from negative to positive at x = a. But $f'(x)$ is itself a function of x which changes sign from negative to positive, therefore, it increases at x = a and hence its derivative $f''(x)$ is positive at x = a. Hence a function y = f(x) has a minimum value at x = a if

(i) $f'(x) = 0$ at x = a

(ii) $f''(x)$ is positive at x = a

REMARKS: (1) Maximum and minimum values are all stationary values, but the converse is not true, that is, a stationary value need not be a maximum or minimum value, because the curve may have a point of inflexion at x = a.

(2) $f'(x) = 0$ at x = a implies that the tangent to the curve y = f(x) at x = a is parallel to the x–axis. Therefore at stationary points the tangent is parallel to x–axis.

Working Rules for Finding the Maximum and Minimum Values of y = f(x)

First method:

(i) Find $f'(x)$ and equate it to zero. Solve this equation for real values of x. Let these values be a, b, c, …

(ii) Find $f''(x)$. Put x = a, b, c, … turn by turn.

If $f''(x)$ is negative at x = a, then f(x) is maximum at x = a and the corresponding maximum value of f(x) is f(a). If $f''(x)$ is positive at x = a, then f(x) is minimum at x = a and the corresponding minimum value of f(x) is f(a). Similarly for the points x = b, c, …

(iii) If $f''(x) = 0$ at x = a but $f'''(x) \neq 0$ at x = a, then x = a is a point of inflexion.

Second method:

Sometimes, the process of finding $f''(x)$ becomes tedious. In such cases, first derivative test should be preferred.

(i) Find $f'(x)$ and equate it to zero. Solve this equation for real values of x. Let these values be a, b, c, …

(ii) Consider the value x = a. Study the signs of $f'(x)$ for values of x slightly less than a and slightly greater than a.

(iii) If $f'(x)$ changes sign from positive to negative then f(x) is maximum at x = a. If $f'(x)$ changes sign from negative to positive then f(x) is minimum at x = a.

(iv) If $f'(x)$ does not change sign, then $x = a$ is a point of inflexion. Similarly for the points $x = b, c, \ldots$

Example 04: Determine the maximum and minimum values of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$.

Solution: Given that $f(x) = x^5 - 5x^4 + 5x^3 - 1$ (1)

Differentiating (1) with respect to x, we get

$$f'(x) = 5x^4 - 20x^3 + 15x^2 \quad (2)$$

$$f'(x) = 0 \Rightarrow 5x^4 - 20x^3 + 15x^2 = 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow \quad x^2(x^2 - x - 3x + 3) = 0 \Rightarrow x^2(x-1)(x-3) = 0$$

Thus, the stationary values are $x = 0, 1, 3$.

Differentiating (2) with respect to x, we get: $f''(x) = 20x^3 - 60x^2 + 30x$ (3)

For $x = 0$, $f''(0) = 0$. Therefore, $x = 0$ gives neither maximum nor minimum value of f(x). Also, $f'''(x) = 60x^2 - 120x + 30$.

For $x = 0$ $f'''(0) = 30 \neq 0$, therefore, $x = 0$ is a point of inflexion.

For $x = 1$, $f''(1) = 20 - 60 + 30 = -10 < 0$, therefore, f(x0 is maximum at $x = 1$. The maximum value at $x = 1$ is: $f(1) = 1 - 5 + 5 - 1 = 0$.

For $x = 3$, $f''(3) = 20(3)^3 - 60(3)^2 + 30(3) = 90 > 0$, therefore, f(x) is minimum at $x = 3$. The minimum value at $x = 3$ is: $f(3) = (3)^5 - 5(3)^4 + 5(3)^3 - 1 = -28$.

Applications of Maxima and Minima

In this section, we shall discuss some of important applications of maxima and minima of a function f(x) from different areas. It may be noted that the many functions whose maximum and minimum values are required are not directly given. These have to be formed from the given data. If function contains two variables, one of them has to be eliminated with the help of condition imposed on them.

Example 05: MUET, Jamshoro advertises for a short course as per ISO requirements. The profit from this course is $P(x) = -0.02x^2 + 120x + 100$ rupees, where x is an amount spent on advertisement. Find the amount to be spent on advertisement in order to maximize the profit. Also find the maximum profit.

Solution: We have $P(x) = -0.02x^2 + 120x + 100$ (1)

Differentiating (1) with respect to x, we get: $P'(x) = -0.04x + 120$

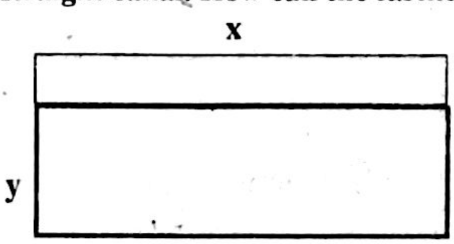
Now $P'(x) = 0 \Rightarrow -0.04x + 120 = 0 \Rightarrow x = 3000$

Also, $P''(x) = -0.04$

At $x = 3000$, $P''(3000) = -0.04 < 0 \Rightarrow P(x)$ is maximum at $x = 3000$.

Thus profit is maximum if the university spends Rs. 3000 on advertisement, and the maximum profit is: $P(3000) = -0.02(3000)^2 + 120(3000) + 100 = \text{Rs. } 180{,}100$.

Example 06: A farmer has 1000 meter of barbed wire which he is to fence off three sides of a rectangular field, the fourth side being bounded by a straight canal. How can the farmer enclose the largest field?



Solution: Let the dimensions of rectangular field be x and y meters respectively. Then the area

$$A = xy \quad (1)$$

According to given conditions, we have

$x + y + y = x + 2y = 1000$ or $x = 1000 - 2y$

Put this in (1), we get: $A = (1000-2y)(y) = 1000y - 2y^2$ (2)

Differentiating (2) with respect to y, we get

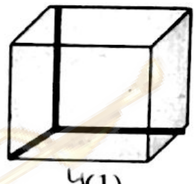
$$dA/dy = 1000 - 4y$$

Now, $dA/dy = 0 \Rightarrow 1000 - 4y = 0 \Rightarrow y = 250 \Rightarrow x = 500$

Also, $d^2A/dy^2 = -4$. At y = 250, we have: $d^2A/dy^2 = -4 < 0$.

Thus area 'A' of field is maximum if its dimensions are 500×250. Then maximum area covered is A = xy = (500) (250) = 125, 000 m^2

Example 07: A topless rectangular box with a square base is having a volume of 1296 cubic cm. The material for the base costs Rs.3 per square cm and the material for the sides cost Rs. 2 per square cm. What dimensions should the box have to minimize its cost and what is the minimum cost?



Solution: Let the length of the topless rectangular box be x cm, width be also x cm (since the base is a square) and height be y cm. Then

$$x \cdot x \cdot y = 1296 \quad (\text{using } V = lwh)$$

or, $$x^2 y = 1296 \quad (1)$$

Cost of material for the base $= 3x^2$ (Rs)

Cost of material for the four sides $= 2(4xy) = 8xy$ (Rs)

Thus total cost of the box is: $C = 3x^2 + 8xy$

From (1), $y = 1296/x^2$. Thus, $C = 3x^2 + 8x\left(\frac{1296}{x^2}\right) = 3x^2 + \frac{10368}{x}$ (2)

This is the function to be minimized. Differentiating (2) with respect to x, we get

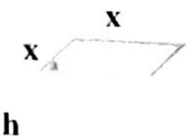
$$C' = 6x - (10368/x^2)$$

For minima, $C' = 0 = 6x - (10368/x^2) \Rightarrow 6x^3 - 10368 = 0 \Rightarrow x = 12$

Now, $C'' = 6 + (20736/x^3)$. At x = 12, we have, $C'' = 18 > 0$.

Thus, C is minimum for x = 12. Substituting this into equation (1), we get y = 9. Hence, the required dimensions of the box are: 12 cm, 12 cm and 9 cm. The minimum cost will be obtained by putting x = 12 in (2) and it will be Rs. 1296/.

Example 08: A manufacturer of storage bins plans to produce some open–top rectangular boxes with square bases. The volume of each box is to be 100 cubic feet. Material for the base costs \$8 per square foot, and material for the sides costs \$5 per square foot. Determine the dimensions of the box that will minimize the cost of materials.



Solution: Suppose x be the width of the bin. Since the base is square, the length must also be x. No information is given about the height, so we will use h to represent it.

The area of the base is $x \times x$ or $x^2 ft^2$. Thus, at \$8 per square foot, the cost of the material for the base is $8x^2$ dollars. The area of each side is $x \times h$, so the area of the four sides is 4xh. Since the cost of the material for the sides is \$5 per square foot, the cost of the material for all four sides is $\$5 \times 4xh$ or 20xh dollars. Thus, the total cost C for all the materials is of a bin is

$$C = 8x^2 + 20xh \text{ dollars}$$ (1)

Since, $V = x \cdot x \cdot h \Rightarrow V = x^2 h \Rightarrow 100 = x^2 h \quad (V = 100 ft^3) \rightarrow h = 100/x^2$

Now C becomes: $C = 8x^2 + 20x\left(\frac{100}{x^2}\right) \Rightarrow C = 8x^2 + \frac{2000}{x}$ (2)

Differentiating (1) with respect x, we get: $C' = 16x - 2000/x^2$

For critical values, we have

$$16x - (2000/x^2) = 0 \Rightarrow 16x^3 - 2000 = 0 \Rightarrow x^3 = 125 \Rightarrow x = 5.$$

Now, $C'' = 16 + \frac{4000}{x^3}$. At x = 5 we get: $C''(5) = 16 + \frac{4000}{(5)^3} = 48 > 0$

Thus, C is minimum at x = 5. The value of h can now be found from

$$h = 100/x^2 = 100/25 = 4$$

We conclude that the base should be made 5 feet by 5 feet and the height should be 4 feet in order to minimize the cost. In this case the minimum cost of the box will be C = 600 dollars. [Use equation (2)]

Example 09: A farmer has 1600 feet of fencing to make a rectangular enclosure for his dogs. What should be the dimensions of the enclosure if he wants the largest area?

Solution: Let x feet and y feet be the length and width of the rectangular pen respectively. Then perimeter is:

$2(x+y) \Rightarrow 1600 = 2(x+y) \Rightarrow y = 800 - x$ (1)

The area $A = xy = x(800 - x) = 800x - x^2$ (2)

Differentiating (2) with respect to x, we get

$$dA/dx = 800 - 2x$$

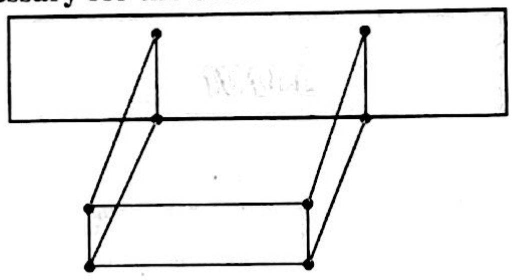
For critical numbers, we have

$$800 - 2x = 0 \Rightarrow x = 400$$

Also, $d^2A/dx^2 = -2$ which is negative.

Thus, area A is maximum when x = 400 feet. Hence, the dimensions of field must be: x = 400 feet and $y = 800 - 400 = 400$ feet to have the maximum area. The maximum area then will be $A = 1600\ ft^2$.

Example 10: A builder decides to fence in a rectangular area of 800 square feet behind his warehouse, using the wall of the building as one of the four sides (see figure). What is the least amount of fencing necessary for the other three sides?



Solution: Let x feet and y feet be the width and length of rectangular area respectively. Then length of fencing

$$z = y + x + x = y + 2x$$

or, $y = z - 2x$ (1)

Also, $\text{Area} = (x)(y) = xy$ (2)

$800 = x(z - 2x)$, from (1)

$$800 = xz - 2x^2 \Rightarrow z = \frac{800 + 2x^2}{x} \Rightarrow z = 2x + \frac{800}{x} \quad (3)$$

Differentiating (3) with respect to x, we get: $\frac{dz}{dx} = 2 - \frac{800}{x^2}$

For critical numbers, we get

$$\frac{dz}{dx} = 2 - \frac{800}{x^2} = 0 \Rightarrow x^2 = 400 \Rightarrow x = 20 \quad (-\text{ve sign is not admissible})$$

Also, $\frac{d^2z}{dx^2}=\frac{1600}{x^3}$. At x = 20, we have $\left.\frac{d^2f}{dx^2}\right|_{x=20}=\frac{1600}{(20)^3}=\frac{1}{5}>0$.

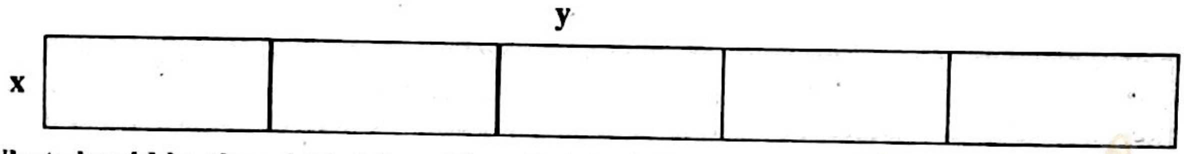
Thus, z is minimum at x = 20. From (2), we have

$$A=xy\Rightarrow 800=20y\Rightarrow y=40$$

Hence, the dimensions of fencing are: 40 feet by 20 feet.

Also, fencing $=40+2(20)=80$ feet. Thus, the least amount of fencing required is 80 feet.

Example 11: A storage company wants to create a storage facility by walling in a rectangular region containing 1728 square feet. It will also use walls to subdivide the region into five equal storage compartments (see the figure).



What should be the width (x) and length (y) of the storage facility in order to use the least amount of material for the walls?

Solution: The area of rectangular field is: $A=xy=1728$ (Given).

Therefore, $y=1728/x$ (1)

The perimeter P = 6x + 2y (Since there are five rooms but six walls)

Or, $P=6x+2y\Rightarrow P=6x+2(1728/x)$ from (1)

Or, $P=6x+3456/x$ (2)

Differentiating (2) with respect to x, we get: $P'(x)=6-3456/x^3$ (3)

For critical numbers, we set

$$P'=0 \Rightarrow 6-3456/x^2=0 \Rightarrow x^2=576 \Rightarrow x=24$$

We need to minimize the P. Therefore,

$$P''(x)=6912/x^3 \quad (4)$$

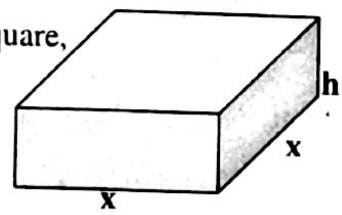
When x = 24: $P''(24)=6912/(24)^3=0.5>0$

Thus, perimeter P is minimum when x = 24 feet ➔ $y=1728/24=72$ [From (1)]

Hence, the dimensions of the storage facility in order to use the minimum material for the walls are: Width = 24 feet and Length = 72 feet.

Example 12: Determine the dimensions of a closed rectangular box with a square base if the volume must be 1000 cubic centimeters and the area of the outside surface is to be as small as possible.

Solution: Let x be the width of the box. Since the base is square, the length must also be x. Let h be its width. So,



$V=x\cdot x\cdot h\Rightarrow V=x^2h$ (1)

But $V=1000$. Thus, $1000=x^2h\Rightarrow h=\frac{1000}{x^2}$ (2)

Also the surface area is,

$S=2Lw+2Lh+2xh\Rightarrow S=2x.x+2x.h+2x.h$

$S=2(x^2+2xh)\Rightarrow S=2\left(x^2+2x\times\frac{1000}{x^2}\right)$ [from (2)]

$$S=2x^2+4000/x \quad (3)$$

Differentiating (3) with respect to x, we get $\frac{dS}{dx} = 4x - \frac{4000}{x^2}$.

For critical numbers, we set

$$4x - \frac{4000}{x^2} = 0 \Rightarrow 4x^3 = 4000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

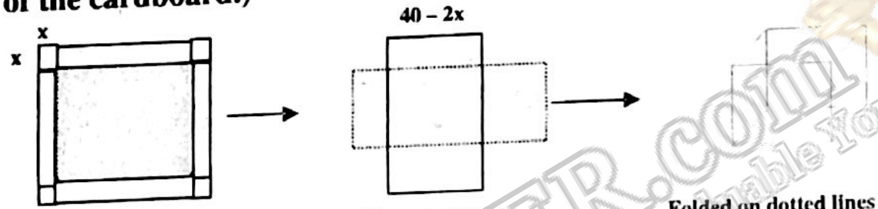
Also, $\frac{d^2S}{dx^2} = 4 + \frac{8000}{x^3}$. At $x = 10$, we have $\frac{d^2S}{dx^2} = 4 + \frac{8000}{10^3} = 12 > 0$.

Thus, S is minimum when x = 10. From (2), we have h = 1000/100 = 10.

Hence, the dimensions of the closed rectangular box are:

Length = 10 cm Width = 10 cm Height = 10 cm

Example 13: A square piece of cardboard 40 centimeters by 40 centimeters is used to make an open box as shown in the figure. A small square is cut from each corner of the cardboard, and then the sides are folded up. Determine the size of the cut (x in the figure) that will lead to the box of largest volume. (Note: At some point it may appear that there are two answers, but only one of them will make sense given the size of the cardboard.)



Original cardboard Corners cut out Folded on dotted lines

Solution: It is given that the length and width of the original cardboard is 40 cm. If a small square is cut from each corner of the cardboard (x in the figure), then length and width of the cardboard will be (40 – 2x) cm. Thus, the volume of the box, that is; after folding, is

$V = (40 - 2x)(40 - 2x)x$, (since x is the height)

$\Rightarrow V = 1600x - 160x^2 + 4x^3$ (1)

We need to maximize the volume V. Therefore, differentiating (1) with respect to x, we get

$$\frac{dV}{dx} = 1600 - 320x + 12x^2$$

For critical numbers, we have

$$V' = 0 \Rightarrow 1600 - 320x + 12x^2 = 0 \Rightarrow 3x^2 - 80x + 400 = 0$$

Using quadratic formula, we have: $x = 20, 20/3$

But we take $x = 20/3$ since x = 20 is not admissible.

Thus, the size of the cut is $x = 20/3$ cm. Also, $\frac{d^2V}{dx^2} d^2V/dx^2 = -320 + 24x$. Put

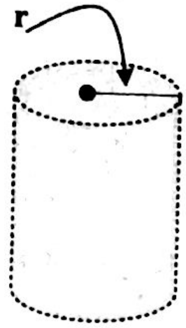
$x = 20/3$, $\left.\frac{d^2V}{dx^2}\right|_{x=\frac{20}{3}} = -320 + 24(20/3) = -160 < 0$. Thus, V is greatest when $x = 20/3$.

Example 14: A metal CAN is to be made in the form of a right circular cylinder that will contain *16π* cubic inches of metal. What radius of the can will require the least amount of metal (see the figure)? Note that there are three parts– a circular top, a circular bottom and the curved side.

Solution: Volume of cylinder is:

$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2} \Rightarrow h = \frac{16\pi}{\pi r^2} \Rightarrow h = \frac{16}{r^2}$ (1)

Surface area $S = 2\pi r^2 + 2\pi rh \Rightarrow S = 2\pi\left(r^2 + r\times\frac{16}{r^2}\right)$, from (1)



$$\Rightarrow S = 2\pi\left(r^2 + \frac{16}{r}\right), \qquad (2)$$

Differentiating (2) with respect to r, we get: $S' = 2\pi\left(2r - \frac{16}{r^2}\right)$.

For critical numbers, we set

$$2\pi\left(2r - \frac{16}{r^2}\right) = 0 \Rightarrow 2r - \frac{16}{r^2} = 0 \Rightarrow 2r^3 = 16 \Rightarrow r^3 = 8 \Rightarrow r = 2.$$

We need to minimize the S. Therefore, $S'' = 2\pi\left(2 + \frac{32}{r^3}\right) \Rightarrow S'' = 4\pi\left(1 + \frac{16}{r^3}\right)$.

When r = 2, $S'' = 4\pi\left(1 + \frac{16}{(2)^3}\right) = 12\pi > 0$.

Thus, S is minimum when radius r = 2 inches. Hence, the can of 2 inches radius will require the least amount of metal.

Example 15: The output power P of a battery is given by $P = VI - RI^2$. For what value of the current I is the power maximum if the voltage V and resistance R in the circuit are 10 volts and 20 ohms respectively?

Solution: Since, $P = VI - RI^2$

Using V = 10 and R = 20, we get

$$P = 10\,I - 20\,I^2 \qquad (1)$$

Differentiating (1) with respect to I, we get

$$dP/dI = V - 2RI \Rightarrow dP/dI = 10 - 40I$$

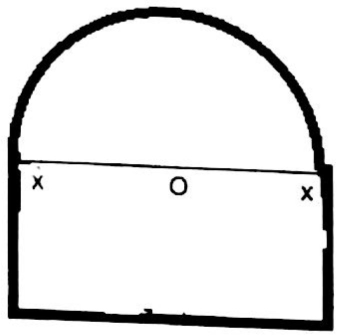
For critical numbers, we set $10 - 40I = 0 \Rightarrow I = 1/4$

Also, $\frac{d^2P}{dI^2} = -40$. When $I = 1/4$, $\left.\frac{d^2P}{dI^2}\right|_{I=1/4} = -40 < 0$

Thus, P is maximum when I = 1/4 A. The maximum power of the battery is then

$$P = 10\,I - 20\,I^2 = 10/4 - 20/16 = (40 - 20)/16 = 1.25 \text{ Watt.}$$

Example 16: A window has the form of rectangle surmounted by a semi-circle. If the perimeter is 40 ft, find its dimensions so that the greatest amount of light may be admitted.



Solution: It may be noted that greatest amount of light is possible when the area of window is maximum.

Now let us look at the figure and problem concerned.

Let x be the radius of the semi-circle so that this side of rectangle is 2x. Let the width of rectangle be y.

The circumference of circle is $2\pi x$ hence the circumference of semi-circle is π x. Therefore the perimeter of entire window is:

$$2x + y + y + \pi x = 40 \text{ (Given)}$$

$$\Rightarrow y = (40 - \pi x - 2x)/2 \qquad (1)$$

[It may be noted that central line is not a part of window. It is merely shown to visualize the center and radius of semi-circle]

Now the area of entire window is:

A = area of rectangle + area of semi-circle

$$A = 2xy + \pi x^2/2 = 2x\left[\frac{40 - \pi x - 2x}{2}\right] + \frac{\pi x^2}{2} = 40x - 2x^2 - \frac{\pi x^2}{2}$$

➔ $$\frac{dA}{dx} = 40 - 4x - \pi x$$

For area to be maximum, dA/dx = 0 ➔ $40 - 4x - \pi x = 0$ ➔ $x = 40/(\pi + 4)$

Now, $d^2A/dx^2 = -4 - \pi$, which is negative. Hence, area of window is maximum if we take $x = 40/(\pi + 4)$. Putting this in (1) and simplifying, we get: $y = 40/(\pi + 4)$.

This shows that to admit maximum light the window should be of square shape. In other words it happens when the radius of semi-circle and with of window are equal.

6.4 DIFFERENTIALS AND THEIR APPLICATIONS

Let $y = f(x)$ ➔ $y + \Delta y = f(x + \Delta x)$

➔ $$\Delta y = f(x + \Delta x) - f(x) \qquad (1)$$

Here Δx denotes a small change in x and Δy the corresponding change in y.

For example, consider: $y = f(x) = x^2 + 2$

If x changes from 1 to 1.1, then corresponding/exact change in the value of y is given by

$$\Delta y = f(1 + 0.1) - f(1) = f(1.1) - f(1) = (1.1)^2 + 2 - (1)^2 - 2 = 0.21$$

Now $\frac{dy}{dx} = f'(x) \quad \Rightarrow dy = f'(x)dx$

If we suppose that $dx = \Delta x$ then $dy = f'(x)\,dx$ will give an approximate change in y. Here dx is called differential of x and the differential of y, written dy, is the product of $f'(x)$ and dx. For example if $y = x^2 + 2$ and x changes from 1 to 1.1 then approximate change in y is: $dy = (2x + 0)\,dx = 2(1)(.1) = 0.2$

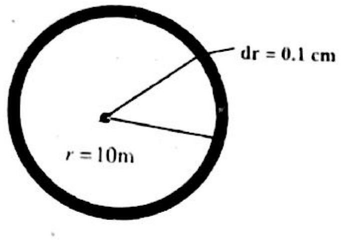
This is approximately equal to 0.21, the exact change in y as shown above. Thus, differentials are useful to obtain approximate change in the dependent variable if the value of independent variable and change in it are known.

Example 01: The radius r of a circle increases from 10 cm to 10.1 cm. Estimate the approximate change in its area. Compare this with the true change.

Solution: The area of circle is given by $A = \pi r^2$

Using differentials, we get

$$dA = \pi(2r)dr \Rightarrow dA = 2\pi r\,dr$$



Substituting the values, we get

$$dA = 2\pi(10m)(0.1m) \Rightarrow dA = 2\pi\ cm^2$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = 2.01\pi = 2\pi + 0.01\pi\ cm^2$$

Thus, approximate error in the computation of area of circle is $0.01\pi\ cm^2$.

Example 02: The radius of a sphere is found by measurement to be 10.5cm with a possible error of 0.1cm. Find the possible error in its surface area and volume.

Solution: Let r, S and V be the radius, surface area and volume of the given sphere respectively. Then: $S = 4\pi r^2 \qquad (1)$

and $V = 4\pi r^3/3 \qquad (2)$

Using differentials, (1) becomes

$$dS = 4\pi(2r)dr = 8\pi r\,dr$$

Substituting the values, we get

$$dS = 8\pi(10.5cm)(0.1cm) = 8.4\pi\ cm^2$$

This is an approximate error in surface area of sphere.
Now using differentials, (2) becomes

$$dV = 4\pi(3r^2)dr/3 = 4\pi r^2 dr$$

Substituting the values, we get: $dV = 4\pi(10.5cm)^2(0.1cm) = 44.1\pi cm^3$

Thus an approximate error in the volume of sphere = V – dV = 1500 π cm³.

Example 03: A spherical balloon has radius 10 units. Show that the percentage increase in its volume is approximately 0.3units, if its radius increases 0.1 percent.

Solution: Let r and V be the radius and volume of the spherical balloon respectively. Then, $V = 4\pi r^3/3$ (1)

Using differentials, (1) becomes: $dV = 4\pi(3r^2)dr/3 = 4\pi r^2 dr$

Substituting the given values, we get

$$dV = 4(3.14)(10)^2(0.01) \quad \left(\text{using } dr = 0.1\% \text{ of } r = 0.1\left(\frac{1}{100}\right)(10) = 0.01\right)$$

➔ $dV = 12.56\ unit^3$

Percentage error: $\frac{dV}{V}\times 100\% = \frac{12.56}{\frac{4}{3}(3.14)(10)^3}\times 100\% = \frac{37.68}{12560}\times 100\% = 0.3\%$.

This shows that percentage increase in the volume of the spherical balloon is 0.3%.

Example 04: Use differentials to approximate the change in volume of a sphere shaped tumor when its radius increases from 1 to 1.1 cm.

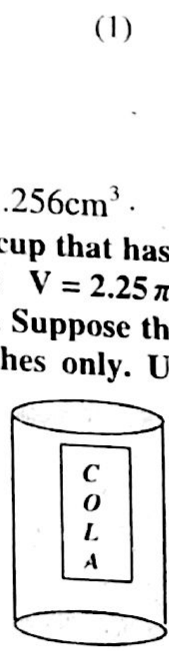
Solution: The volume of sphere is: $V = 4\pi r^3/3$ (1)

Using differentials, (1) becomes: $dV = 4\pi(3r^2)dr/3 = 4\pi r^2 dr$

Substituting the values, we get: $dV = 4(3.14)(1)^2(0.1) = 1.256cm^3$

Thus, approximately the change in volume of the sphere–shaped tumor is $1.256cm^3$.

Example 05: A fast food restaurant serves soft drinks in a cylindrical cup that has a radius of 1.5 inches. The volume of soft drink that this cup can hold is cubic inches. Ordinarily, the restaurant fills up to a height of 8inches. Suppose that instead the restaurant decides to fill the cup to a height of 7.5 inches only. Use differentials to approximate the number of cubic inches of soft drink the restaurant saves on each serving. $V = 2.25\pi h$



Solution: Since $V = 2.25\pi h$ (1)

Using differentials, (1) becomes: $dV = 2.25\pi dh$

Substituting the given values, we get

$$dV = 2.25(3.14)(-0.5) = -3.53\ in^3.$$

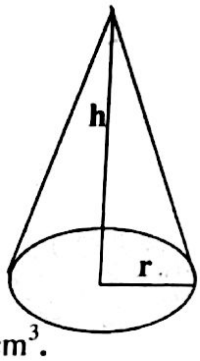
Here, negative sign shows the decrease in capacity of the cup. Hence, approximately the number of cubic inches of the soft drink the restaurant saves on each serving is 3.53 in³.

Example 06: Leaking sand forms a conical pile in which the height is always twice the radius (h = 2r). Consider the moment at which the radius is 9cm. Use differentials to determine the approximate change in the volume when the radius changes by 0.1 cm.

Solution: The volume of cone is $V = \pi r^2 h/3$ (1)

It is given that the height is twice the radius, therefore h = 2r

➔ $V = \pi r^2(2r)/3 \Rightarrow V = 2\pi r^3/3$ (2)



Using differentials, (2) becomes

$$dV = 2\pi\left(3r^2\right)dr/3 = 2\pi r^2 dr$$

Substituting the values, we get: $dV = 2(3.14)(9)^2(0.1) = 50.868 \text{ cm}^3$

Hence, the approximate change in the volume of the conical pile is 50.868 cm^3.

WORKSHEET 06

1. Find the equations of tangent and normal to the curves (a) $x^2 - xy + y^2 = 7$ at the point (-1, 2) (b) y(x – 2) (x – 3) – x + 7 = 0 at the point where it cuts the x-axis.

2. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x – axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

3. At what points of the curve $y = 2x^3 - 3x^2 - 2x + 4$ are the tangents parallel to the line $10x - y + 7 = 0$? Find the equation of the normal at each of these points. Also find the angle between the two curves: $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

4. Find the angle between two curves: $x^2 - y^2 = a^2$ and $x^2 + y^2 = \sqrt{2}\,a^2$

5. Show that parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.

6. Prove that curves $x^2/a^2 + y^2/b^2 = 1$ and $x^2/c^2 + y^2/d^2 = 1$ will cut orthogonally if a – b = c – d.

7. Find the length of tangent, normal, sub-tangent and sub-normal to the curve:

$$x = a(t + \sin t),\ y = a(1 - \cos t)$$

8. For the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, show that length of tangent is $y \operatorname{cosec}\theta$. Also find the length of: sub-tangent, normal and sub-normal.

9. For the curve $x = a(\ln\cot\theta/2 - \cos\theta)$, $y = a\sin\theta$, find the lengths of subtangent and sub-normal at the point $\theta = \pi/4$

10. Find the radius of curvature at the given point of each of the following curves:

(i) $y = x^2 - 5x - 6;\ (4, -10)$ (ii) $9x^2 + 16y^2 = 180;\ (2, 3)$

(iii) $y = \sec x;\ \left(\frac{\pi}{3}, 2\right)$ (iv) $x^3 + y^3 = 3axy;\ \left(\frac{3a}{2}, \frac{3a}{2}\right)$

11. Find the radius of curvature of the given curve at the specified point:

(i) $r = 4\sin 2\theta;\ \left(2, \frac{1}{12}\pi\right)$ (ii) $r = a\sec 2\theta;\ (a, 0),\ a > 0$

12. Find the equation of the osculating circle to the curve y = ln x at the point (1, 0).

13. Find the equation of the osculating circle to the curve $x^2/4 - y^2/9 = 1$ at the point (-2, 0).

14. A tennis ball thrown straight up is $-16t^2 + 96t + 7$ feet above the ground after t seconds. How high will the ball go?

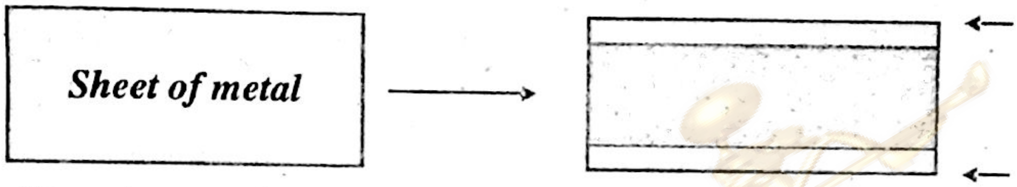
15. The sum of two positive numbers is 100. If their product is a maximum, what are the two numbers? (Hint: Let one number be x. What expression represents the other number?)

16. A farmer has 1600 feet of fencing to make a rectangular pen for his hogs. What should be the dimensions of the pen if he wants the largest area?

17. What is the smallest amount of fencing that can be used to enclose a rectangular garden having an area of 900 square feet?

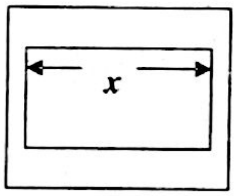
18. An open rectangular box (that is, a box with no top) with a capacity of 36,000 cubic inches is needed. If the box must be twice as long as it is wide, what dimensions would require the least material?

19. A builder plans to construct a gutter from a long sheet of metal by making two folds of equal size (see the figure). The folds are made to create perpendicular sides.

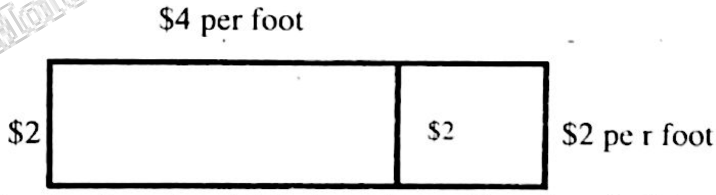


The metal is 28 centimeters wide and 500 centimeters long. How much (x) should be turned up for each side in order for the gutter to hold the most water?

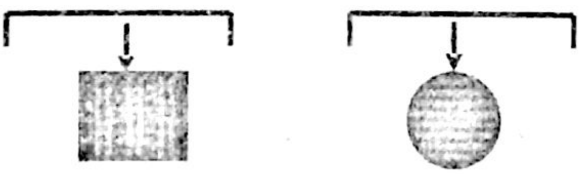
20. Determine the dimensions of the smallest size (that is, smallest area) rectangular piece of paper that satisfies all of the following conditions:
(a) You can print 50 square inches of material on it (the shaded area in the fig).
(b) There will be 2 – inch margins on the top and bottom.
(c) There will be 1 – inch margins on the sides.



21. A gardener wishes to fence in a rectangular area of 1728 square feet. He also wants to insert a fence that will divide the area into two rectangular sub areas. The drawing shows that some fencing costs $4 per foot and some costs $2 per foot. Find the dimensions that will minimize the cost of the fencing.



22. A wire 50 centimeters long is cut into two pieces. One piece (call its length x) will be bent to form a square. The other piece (of length $50 - x$) will be bent to form a circle. How much wire should be used for the square if the total area (square plus circle) is to be the smallest possible?



23. Test the curve $y = x^4$ for point of inflexion.

24. Find a and b if the function $f(x) = a/x + bx$ such that $f(2) = 1$ possesses extreme point at $x = 2$. Is (2, 1) a point of minima or maxima?. Confirm it by drawing the graph.

25. Show that function $f(x) = \sin^m x \cos^n x$ attains a maximum when $x = \tan^{-1}(m/n)$

26. The horse power developed by an aircraft traveling horizontally with velocity v feet per second is given by $H = aw^2/v + bv$, where a,b, w are constants. For what value of v is the horse power H is maximum?

27. The velocity of waves of wave length λ on deep water is proportional to $\sqrt{\lambda / a + a / \lambda}$, where a ia some constant. Prove that the velocity is maximum when $\lambda = a$.

28. In a submarine telegraph cable, the speed of signaling varies as $x^2 \ln(1/x)$, where x is the ratio of radius of the core to that of the covering. Show that greatest speed is attained when x is $1/\sqrt{e}$.

29. The efficiency e of a screw jack is given by $e = \tan x/\tan(x + a)$, where a is a constant. Find x if this efficiency is to be maximum. Also find the maximum efficiency.

30. Show that all the rectangles of given area, the square has the least parameter.

31. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.

32. After t hours, the number of bacteria in a laboratory culture id given by $n = 6t^2 + 200$. Use differentials to approximate the change in the number of bacteria when t changes from 5 hours to 5 hours and 3 minutes.

33. Use differentials to approximate the change in volume of a sphere – shaped tumor $\left(V = 4/3\pi r^3\right)$ when its radius increases from 1 to 1.1 centimeters.

34. Leaking sand forms a conical pile in which the height is always thrice the radius (h = 3r). Consider the moment when the radius is 9cm. Use differentials to determine the approximate change in the volume when the radius changes by 1%, and volume of the cone is given by $V = \pi r^2 h/3$.

35. A fast food restaurant serves soft drinks in a cylindrical cup that has a radius of 1.5 inches. The volume of soft drink that this cup can hold is given by $V = \pi r^2 h$. Ordinarily, the restaurant fills up to a height of 7.8 inches. Suppose that instead the restaurant decides to fill the cup to a height of 8inches. Use differentials to approximate the number of cubic inches of soft drink the restaurant spend on each serving.

CHAPTER SEVEN

CALCULUS OF SEVERAL VARIABLES

7.1 INTRODUCTION

The idea of several variables was introduced in Chapter 4. The concept of partial derivatives of first and higher derivatives was also discussed.

In this chapter, we shall study other properties of partial derivatives such as homogeneous function and Euler's Theorem such as differentials, extreme values in two variables.

Homogeneous Functions

A function f(x, y) is called a homogeneous function of degree n if it can be expressed in the form $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. For instance,

1. Consider $f(x,y)=\dfrac{x^3+y^3}{x-y}$. Replacing x by λ x and y by λ y, we get

$$f(\lambda x,\lambda y)=\frac{\lambda^3x^3+\lambda^3y^3}{\lambda x-\lambda y}=\frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}=\lambda^2\frac{x^3+y^3}{x-y}=\lambda^2 f(x,y).$$

Thus, f(x, y) is a homogeneous function of degree 2.

2. Let $f(x,y)=\dfrac{x}{y}+\dfrac{3}{4}\dfrac{y}{x}+\cos\sqrt{\dfrac{y}{x}}+\ln x-\ln y$.

Replacing x by λ x and y by λ y, we get

$$f(\lambda x,\lambda y)=\frac{\lambda x}{\lambda y}+\frac{3}{4}\frac{\lambda y}{\lambda x}+\cos\sqrt{\frac{\lambda y}{\lambda x}}+\ln\lambda x-\ln\lambda y=\frac{x}{y}+\frac{3}{4}\frac{y}{x}+\cos\sqrt{\frac{y}{x}}+\ln\lambda+\ln x-\ln\lambda-\ln y$$

$$f(\lambda x,\lambda y)=\frac{x}{y}+\frac{3}{4}\frac{y}{x}+\cos\sqrt{\frac{y}{x}}+\ln x-\ln y=f(x,y)=\lambda^0 f(x,y).$$

Thus, f(x, y) is a homogeneous function of degree 0.

3. Let $f(x,y)=\dfrac{\sqrt{y}+\sqrt{x}}{y+x}$.

Replacing x by λ x and y by λ y, we get

$$f(\lambda x,\lambda y)=\frac{\sqrt{\lambda y}+\sqrt{\lambda x}}{\lambda y+\lambda x}=\frac{\sqrt{\lambda}\left(\sqrt{y}+\sqrt{x}\right)}{\lambda(y+x)}=\lambda^{-1/2}\frac{\sqrt{y}+\sqrt{x}}{y+x}=\lambda^{-1/2}f(x,y).$$

Thus, f(x, y) is a homogeneous function of degree $-1/2$.

4. Let $f(x,y)=\sin\dfrac{x^3+y^3}{x-y}$. This function is not homogeneous. However, if we let

$$f(x,y)=\sin\left(\frac{x^3+y^3}{x-y}\right)=u \quad\Rightarrow \sin^{-1}u=\frac{x^3+y^3}{x-y}=z$$

Replacing x by λx and y by λy, we get

$$z = g(\lambda x, \lambda y) = \frac{\lambda^3 x^3 + \lambda^3 y^3}{\lambda x - \lambda y} = \left[\frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}\right] = \lambda^2\left(\frac{x^3+y^3}{x-y}\right) = \lambda^2 g(x, y).$$

Therefore, $z = g(x, y)$ is a homogeneous function.

5. Let $f(x, y) = \ln x - \ln y + \frac{x+y}{x-y}$.

Replacing x by λx and y by λy, we get

$$f(\lambda x, \lambda y) = \ln \lambda x - \ln \lambda y + \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \ln \lambda + \ln x - \ln \lambda - \ln y + \frac{\lambda(x+y)}{\lambda(x-y)}$$

$$f(\lambda x, \lambda y) = \ln x - \ln y + \frac{x+y}{x-y} = f(x, y) = \lambda^0 f(x, y).$$

Thus, $f(x, y)$ is a homogeneous function of degree 0.

Definition: If $f(x, y)$ is a homogeneous function of degree n, it can be expressed as:

$$f(\lambda x, \lambda y) = \lambda^n g(y/x)$$

For example, consider the function $f(x, y) = \frac{x^3+y^3}{x-y}$ which is a homogeneous function of degree 2. Now

$$f(x, y) = \frac{x^3+y^3}{x-y} = \frac{x^3(1+(y/x)^3)}{x(1+y/x)} = x^2 g\left(\frac{y}{x}\right) \text{ where } g\left(\frac{y}{x}\right) = \frac{(1+(y/x)^3)}{(1+y/x)}$$

7.2 EULER'S THEOREM

Statement: If $u = f(x, y)$ is a homogeneous function of degree n, in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Proof: Since $u = f(x, y)$ is a homogeneous function of degree n in x and y, therefore it can be expressed as: $u = x^n f\left(\frac{y}{x}\right)$.

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = nx^{n-1} \cdot f\left(\frac{y}{x}\right) + x^n \cdot f'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = nx^{n-1} \cdot f\left(\frac{y}{x}\right) + x^n \cdot f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right).$$

$$\frac{\partial u}{\partial x} = nx^{n-1} \cdot f\left(\frac{y}{x}\right) - x^{n-2} y \cdot f'\left(\frac{y}{x}\right)$$

Multiplying both sides by x, we get

$$x\frac{\partial u}{\partial x} = nx^n \cdot f\left(\frac{y}{x}\right) - x^{n-1} y \cdot f'\left(\frac{y}{x}\right) \Rightarrow x\frac{\partial u}{\partial x} = nu - x^{n-1} y \cdot f'\left(\frac{y}{x}\right) \qquad (1)$$

Now differentiating u partially w.r.t y, we get

$$\frac{\partial u}{\partial y} = (0) \cdot f\left(\frac{y}{x}\right) + x^n \cdot f'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = x^n \cdot f'\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right) = x^{n-1} \cdot f'\left(\frac{y}{x}\right)$$

Multiplying both sides by y, we get: $y\frac{\partial u}{\partial y} = x^{n-1} y \cdot f'\left(\frac{y}{x}\right)$ (2)

Adding (1) and (2), we obtain: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu - x^{n-1} \cdot f'\left(\frac{y}{x}\right) + x^{n-1} y \cdot f'\left(\frac{y}{x}\right)$

Therefore, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu\cdot$$

Note: Euler's Theorem can be extended to a homogeneous function of any number of variables. Thus, if $f\left(x_1,x_2,x_3,\cdots,x_n\right)$ be a homogeneous function of n variables $x_1,x_2,x_3,\ldots,x_n$ of degree n, then: $x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\ldots+x_n\frac{\partial f}{\partial x_n}=nf$

Example 01: Verify Euler's Theorem for the functions

(i) $f(x, y) = ax^2 + bxy + cy^2$ **(ii)** $f(x,y)=\frac{x^{3/4}+y^{3/4}}{x^{1/2}+y^{1/2}}$

Solution: (i) We observe that u = f(x, y) is homogeneous function of degree 2 hence by Eule's Theorem we have to show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\cdot$

Now, $$\frac{\partial u}{\partial x}=2ax+by \quad \rightarrow \quad x\frac{\partial u}{\partial x}=2ax^2+bxy \qquad (1)$$

Also, $$\frac{\partial u}{\partial y}=2bx+2cy \quad \rightarrow \quad y\frac{\partial u}{\partial x}=2bxy+2cy^2 \qquad (2)$$

Adding (1) and (2), we get: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2ax^2+2bxy+2cy^2=2\left(ax^2+bxy+cy^2\right)=2u$

This verifies Euler's Theorem.

(ii) We observe that u = f(x, y) is homogeneous function of degree (3/4 – 1/2) =1/4, hence by Euler's Theorem we have to show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{4}u$

Now, $$\frac{\partial u}{\partial x}=\frac{\left(x^{1/2}+y^{1/2}\right)\left(3/4\ x^{-1/4}\right)-\left(x^{13/4}+y^{3/4}\right)\left(1/2\,x^{-1/2}\right)}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$\rightarrow \quad x\frac{\partial u}{\partial x}=\frac{\left(x^{1/2}+y^{1/2}\right)\left(3/4\ x^{3/4}\right)-\left(x^{13/4}+y^{3/4}\right)\left(1/2\,x^{1/2}\right)}{\left(x^{1/2}+y^{1/2}\right)^2} \qquad (1)$$

Also, $$\frac{\partial u}{\partial y}=\frac{\left(x^{1/2}+y^{1/2}\right)\left(3/4\ y^{-1/4}\right)-\left(x^{13/4}+y^{3/4}\right)\left(1/2\,y^{-1/2}\right)}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$\rightarrow \quad y\frac{\partial u}{\partial x}=\frac{\left(x^{1/2}+y^{1/2}\right)\left(3/4\ y^{3/4}\right)-\left(x^{13/4}+y^{3/4}\right)\left(1/2\,y^{1/2}\right)}{\left(x^{1/2}+y^{1/2}\right)^2} \qquad (2)$$

Adding (1) and (2), we get:

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{(3/4)\left[\left(x^{3/4}+y^{3/4}\right)\left(x^{1/2}+y^{1/2}\right)\right]-(1/2)\left[\left(x^{3/4}+y^{3/4}\right)\left(x^{1/2}+y^{1/2}\right)\right]}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$=\frac{\left[\frac{3}{4}-\frac{1}{2}\right]\left[\left(x^{3/4}+y^{3/4}\right)\left(x^{1/2}+y^{1/2}\right)\right]}{\left(x^{1/2}+y^{1/2}\right)^2}=\frac{1}{4}\left[\frac{\left(x^{3/4}+y^{3/4}\right)}{\left(x^{1/2}+y^{1/2}\right)}\right]=\frac{1}{4}u$$

This verifies Euler's Theorem.

Example 02: If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ **show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Solution: Given that: $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \Rightarrow \tan u=\frac{x^3+y^3}{x-y}$.

Let $$z=\tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)}=x^2\frac{1+(y/x)^3}{1-y/x}.$$

Thus z is a homogeneous function of degree 2 in x and y. Thus by Euler's theorem we have: $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad (1)$$

Substituting the value of z in (1), we get

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2(\tan u)\Rightarrow x(\sec^2 u)\left(\frac{\partial u}{\partial x}\right)+y(\sec^2 u)\left(\frac{\partial u}{\partial y}\right)=2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2\tan u\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sin u}{\cos u}\times\cos^2 u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \quad (\text{using } \sin 2\theta=2\sin\theta\cos\theta).$$

Example 03: If $u=ln\left(\frac{x^2+y^2}{x+y}\right)$, **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$.

Proof: Given that: $u=\ln\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow e^u=\frac{x^2+y^2}{x+y}$

Let $$z=e^u=\frac{x^2+y^2}{x+y}=\frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}=x\frac{1+(y/x)^2}{1+y/x}.$$

Thus z is a homogeneous function of degree 1 in x and y. Thus by Euler's theorem we have: $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z \qquad (1)$$

Substituting the value of z in (1), we get

$$x\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=e^u\Rightarrow x(e^u)\left(\frac{\partial u}{\partial x}\right)+y(e^u)\left(\frac{\partial u}{\partial y}\right)=e^u$$

or $$e^u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=e^u\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.$$

Example 04: If ln u = [(x³ + y³)/(3x + 4y)], show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\ \ln u$.

Solution: Let us assume that: $z=\ln u \qquad (1)$

$\Rightarrow$ $$z=[(x^3+y^3)/(3x+4y)] \qquad (2)$$

From (2) we see that z is a homogeneous function of degree 2. Hence by Euler's Theorem: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$ (3)

Now from (1), $\frac{\partial z}{\partial x}=\frac{1}{u}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial x}=\frac{1}{u}\frac{\partial u}{\partial x}$. Substituting these values in (3), we get:

$$x\frac{1}{u}\frac{\partial u}{\partial x}+y\frac{1}{u}\frac{\partial u}{\partial y}=2\ln u \quad \Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\ln u$$

Example 05: If $u=\sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$ **show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-7\tan u$

Solution: Let, $z=\frac{x+2y+3z}{x^8+y^8+z^8}$ (1)

$\Rightarrow$ $u=\sin^{-1}z$ $\Rightarrow z=\sin u$ (2)

From (1) we observe that z is a homogeneous function of degree -7. Thus by Euler's Theorem: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-7z$ (3)

Now from (2), $\frac{\partial z}{\partial x}=\cos u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\cos u\frac{\partial u}{\partial y}$. Substituting these in (3), we get

$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=-7\sin u$. Dividing by cos u, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-7\tan u$$

Example 06: If U = f(x, y) is a homogeneous function of degree n, prove that
$x^2f_{xx}+2xy\,f_{xy}+y_2\,f_{yy}=n(n-1)\,f(x,y)$

Solution: Since f is a homogeneous function of degree 10, we have

$$xf_x+yf_y=nf \quad (1)$$

Differentiating (1) with respect to x, we get

$$xf_{xx}+f_x(1)+yf_{yx}=nf_x \Rightarrow xf_{xx}+f_x+yf_{yx}=nf_x \quad (2)$$

Differentiating (1) with respect to y, we get

$$xf_{xy}+yf_{yy}+f_y(1)=nf_y \Rightarrow xf_{xy}+yf_{yy}+f_y=nf_y \quad (3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$x\left(xf_{xx}+f_x+yf_{xy}\right)+y\left(xf_{xy}+yf_{yy}+f_y\right)=nxf_x+nyf_y$$

$$\Rightarrow \quad x^2f_{xx}+xf_x+xyf_{xy}+xyf_{xy}+y^2f_{yy}+yf_y=n\left(xf_x+yf_y\right)$$

$$x^2f_{xx}++xyf_{xy}+xyf_{xy}+y^2f_{yy}+\left(xf_x+yf_y\right)=n\left(xf_x+yf_y\right)$$

$$x^2f_{xx}+2xyf_{xy}+y^2f_{yy}+nf=n(nf)\Rightarrow x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n^2f-nf$$

or $$x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f$$

7.3 TOTAL DIFFERENTIALS

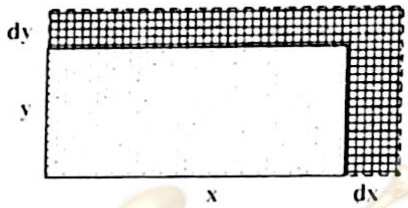
We have studied the concept of the differentials of a function of one variable. For $y=f(x)$ the differential was defined as $dy=f'(x)dx$ with $dx=\Delta x$ and $dy\approx\Delta y$. The differential dy was used to approximate Δy for small change dx in x. We estimated such as the change in revenue associated with small changes in advertising expenditures and the change in price that would cause a small change in demand. Similarly, the change in volume of cone may occur due to change in its radius and height. The differential concept can be extended to functions of two or more variables.

Definition: Let $z = f(x, y)$. The total differential dz is defined as:

$$dz = f_x(x, y)dx + f_y(x, y)dy \qquad \text{or,} \qquad dz = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy,$$

where $dx = \Delta x, dy = \Delta y$ and $dz \approx \Delta z$ and $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.

Example 01: A rectangular plate expands in such a way that its length changes from 10 cm to 10.5 cm and its breadth changes from 5 to 5.3 cm. Find the approximate change in its area.



Solution: Let x and y be the length and breadth of the rectangular plate respectively. According to the question, we have

$x = 10$ cm, $dx = 0.5$ cm, $y = 5$ cm and $dy = 0.3$ cm.

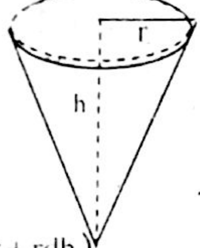
Now area $A = x.y$. Using total differentials, we get

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy \Rightarrow dA = ydx + xdy$$

Substituting the values, we get: $dA = (5)(0.5) + (10)(0.3) \Rightarrow dA = 5.5cm^2$

This is the required change in the area of rectangle plate.

Example 02: A manufacturer of paper drinking cups decides to make its standard cup slightly smaller than before. The cups are conical and hold volume $V = \pi r^2 h/3$ where r is the radius of the top and h is the height (see figure). If the radius is changed from 1.5 inches to 1.4 inches, and the height is changed from 4 inches to 3.9 inches, use differentials to approximate the reduction in volume that results from these changes.



Solution: We have $r = 1.5in, dr = -0.1in, h = 4in, dh = -0.1$ in.

and $\qquad V = \pi r^2 h/3 \qquad (1)$

Using total differentials, (1) becomes

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh \Rightarrow dV = \frac{1}{3}\pi(2rh)(dr) + \frac{1}{3}\pi r^2(dh) \Rightarrow dV = \frac{1}{3}\pi r(2hdr + rdh)$$

Substituting the given values, we get

$$dV = \frac{1}{3}(3.14)(1.5)(2\times 4\times -0.1 + 1.5\times -0.1) = 1.57(-0.8 - 0.15) \Rightarrow dV = -1.4915.$$

The negative sign indicates the reduction in volume. Thus, approximately 1.4915 in^3 volume of soft drink is reduced per cup

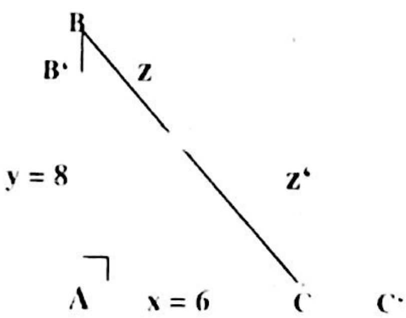
Example 03: Approximate the change in the hypotenuse of a right triangle of legs 6 and 8 units, when the shorter leg is increased by 1/2 units, the longer leg is decreased by 1/4 units.

Solution: Let x and y be the length of shorter and long leg respectively. According to the conditions, we have

$x = 6, dx = 0.5, y = 8$ and $y = -0.25$

Since, $z^2 = x^2 + y^2$ **(Pythagorus theorem)** $\qquad (1)$

$z^2 = (6)^2 + (8)^2 \Rightarrow z^2 = 100 \Rightarrow z = 10$ units



Using total differentials, (1) gives

$2z\,dz = 2x\,dx + 2y\,dy \rightarrow z\,dz = x\,dx + y\,dy$

Substituting the values, we get

$10\,dz = 6(0.5) + 8(-0.25) \Rightarrow 10dz = 3 - 2$

$\Rightarrow dz = 1/10 \Rightarrow dz = 0.1$ unit.

Hence, the approximate change in the hypotenuse is 0.1unit.

[REMARK: ABC is the position of original right triangle and AB'C' is its position after the changes occur in its sizes.]

7.4 MAXIMA AND MINIMA OF A FUNCTION OF TWO VARIABLES

We began our search for extreme values of functions of one variable by finding critical numbers. Now with functions of two variables, we will seek critical points (a, b) since the values x and y need to be maximized or minimized a function $z = f(x, y)$. The definition given next is an extension of the definition of critical number in case of functions of a single variable.

Critical Point

Let $z = f(x, y)$ be defined at point (a, b). Then (a, b) is a critical point of f(x, y) if $f_x(a,b) = 0$ and $f_y(a,b) = 0$. With functions of one variable, the derivative is the slope of the tangent line. With functions of two variables, there are two tangent lines to be considered. The partial derivatives f_x and f_y are the slopes of these two tangent lines. For (a, b) to be a critical point, we insist that f_x and f_y both be zero at (a, b). There must be two horizontal lines at (a, b) one parallel to the x-axis and the other one is parallel to the y-axis.

Example 01: Find the critical points of $f(x, y) = 3x^3 + y^2 - 6x + 2y$.

Solution: First, we find the partial derivatives: $f_x(x,y) = 9x^2 - 36, f_y(x,y) = 2y - 10$.

Next we solve the system of equations created by setting each partial derivative equal to zero. In other words, solve the system

$$\begin{cases} 9x^2 - 36 = 0 & (1) \\ 2y - 10 = 0 & (2) \end{cases}$$

Since each equation contains only one variable, the solution is readily obtained. From the first equation, we have $x = \pm 2$ and from second we have $y = 5$. Thus the critical points are (2, 5) and (-2, 5).

Now we will study the rules to check that whether a critical point is maximum or minimum. A critical point that is neither maximum nor minimum is called the saddle point for $z = f(x, y)$. There is a second derivative test to determine whether a critical point is associated with a maximum, minimum or neither.

Second Partial Derivative Test

For $z = f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then consider $D = (f_{xx})(f_{yy}) - (f_{xy})^2$ to be evaluated at (a, b).

(i) If $f_{xx} < 0$ and $D > 0$, then $(a,b) \Rightarrow$ is a point of maxima.

(ii) If $f_{xx} > 0$ and $D > 0$, then $(a,b) \Rightarrow$ is a point of minima.

(iii) $D < 0$, then $(a,b) \Rightarrow$ is a point of inflexion.

(iv) $D = 0$, then test is inconclusive.

The expression $(f_{xx})(f_{yy}) - (f_{xy})^2$ is called the **discriminator** of f. It is sometimes easier to remember the determinant form: $(f_{xx})(f_{yy}) - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

Example 02: Find the maximum and minimum values of each of the following functions, if there are any.

(i) $f(x, y) = -x^2 + xy - y^2 - 2x - 2y + 3$ (ii) $f(x, y) = x^2 + 3xy - y^2 + 4y - 6x + 1$

Solution: (i) We have $f(x,y) = -x^2 + xy - y^2 - 2x - 2y + 3$

➔ $$f_x(x,y) = -2x + y - 2, \quad f_y(x,y) = x - 2y - 2$$

Now, for extreme values, we must have: $f_x = f_y = 0$

➔ $$\begin{cases} -2x + y - 2 = 0 & (1) \\ x - 2y - 2 = 0 & (2) \end{cases}$$

Solving (1) and (2), we get: x = -2, and y = -2. Thus, (-2, 2) is the only critical point for the given function.

Now, $\quad f_{xx} = -2, f_{yy} = -2$, $f_{xy} = 1$ and $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

At (-2, 2): $\quad D = (-2)(-2) - (1)^2 = 3$

Thus, D > 0 and $f_{xx}(-2,-2) = -2 < 0$. Therefore, (-2, 2) is a maximum point for this function. The maximum functional value is:

$$f(-2,-2) = -(-2)^2 + (-2)(-2) - (-2)^2 - 2(-2) - 2(-2) + 3 = 7.$$

(ii) We have $f(x,y) = x^2 + 3xy - y^2 + 4y - 6x + 1$

➔ $$f_x(x,y) = 2x + 3y - 6, f_y(x,y) = 3x - 2y + 4$$

Now, for extreme values, we must have $f_x = f_y = 0$

➔ $$\begin{cases} 2x + 3y - 6 = 0 & (1) \\ 3x - 2y + 4 = 0 & (2) \end{cases}$$

Solving (1) and (2), we get: x = 0 and y = 2.

Thus, (0, 2) is the only critical point for the given function.

Now, $\quad f_{xx} = 2, f_{yy} = -2$ and $f_{xy} = 3$ and $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

At (0, 2): $\quad D = f_{xx}(0,2) \cdot f_{yy}(0,2) - \{f_{xy}(0,2)\}^2 = (2)(-2) - (3)^2 = -13.$

Since, D < 0, therefore (0, 2) is a saddle point for the given function.

Example 03: A local company advertises on the radio and in the newspaper. Let x and y represent the amounts (in thousands of dollars) spent on the radio and newspaper advertising, respectively. The company's profit based on this advertising has been determined to be (in thousands of dollars) $P(x, y) = -2x^2 - xy - y^2 + 8x + 9y + 10$. How much money should the company spend on each type of advertising in order to maximize the profit?

Solution: We have $\quad P(x,y) = -2x^2 - xy - y^2 + 8x + 9y + 10 \quad (1)$

Differentiating (1) with respect x and with respect to y partially, we get

$$P_x(x,y) = -4x - y + 8, \; P_y(x,y) = -x - 2y + 9$$

Now, for extreme values, we must have $P_x = P_y = 0$

$$\begin{cases} -4x - y + 8 = 0 & (2) \\ -x - 2y + 9 = 0 & (3) \end{cases}$$

Solving (2) and (3), we get: x = 1 and y = 4. Thus, (1, 4) is the only critical point for the given profit function. To apply the second partials test, we must evaluate D for the point

(1, 4). Here, $P_{xx} = -4$, $P_{yy} = -2$, P_{xy} -1 and $D = (f_{xx})(f_{yy}) - (f_{xy})^2$

At (1, 4): $D = (-4)(-2) - (-1)^2 = 7$

Thus, $D > 0$ and $P_{xx}(1, 4) < 0$. Therefore (1, 4) is maximum point for the given function. Now (1, 4) implies that the company should spend \$1000 on the radio and \$4000 on newspaper to get a maximum profit which will be:

$$P(1, 4) = -2(1)^2 - (1)(4) - (4)^2 + 8(1) + 9(4) + 10 = 32$$

This means if company spends \$1000 for radio and \$4000 on TV advertisement the maximum profit will be \$32, 000.

Example 04: Find the three positive numbers that satisfy both of these conditions:

(i) Their sum is 27.

(ii) The sum of the squares of the numbers is as small as possible.

Solution: Let x, y and z be the three required positive numbers. According to the first condition, we have

$$x + y + z = 27 \text{ or } z = 27 - x - y \qquad (1)$$

According to the second condition, we have

$$f(x, y) = x^2 + y^2 + z^2 \qquad (2)$$

Substituting the value of z from (1) into (2), we have

$$f(x, y) = x^2 + y^2 + (27 - x - y)^2 = x^2 + y^2 + (27 - x - y)(27 - x - y)$$

$$f(x, y) = 2x^2 + 2y^2 + 2xy - 54x - 54y + 729 \qquad (3)$$

Differentiating (3) with respect to x and with respect to y partially, we get

$$f_x(x, y) = 4x + 2y - 54, \quad f_y(x, y) = 4y + 2x - 54$$

Now, for extreme values, we must have $f_x = f_y = 0$

$$\begin{cases} 4x + 2y - 54 = 0 & (4) \\ 4y + 2x - 54 = 0 & (5) \end{cases}$$

Solving (4) and (5), we get; x = 9 and y = 9. Thus, (9, 9) is the only critical point for the function. To apply the second partials test, we must evaluate D for the point (9, 9). Here

$$f_{xx} = 4, f_{yy} = 4, f_{xy} = 2 \text{ and } D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

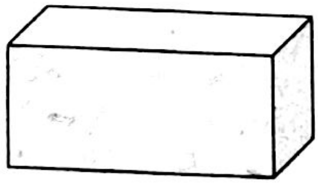
At (9, 9): $D = (4)(4) - (2)^2 = 12$

Thus, $D > 0$ and $f_{xx} > 0$. Therefore (9, 9) is the minimum point for the given function. Substituting x = y = 9 into (1), we have $z = 27 - 9 - 9 = 9$.

Hence, the three required positive numbers are: 9, 9, 9.

Example 05: A rectangular cardboard box (with a top) is being made to contain a volume of 27 cubic feet. Find the dimensions that will minimize the amount of material used to make the box.

Solution: Let the base of the box be x cm by y cm and its height be z cm. Then



$$xyz = 27 \Rightarrow z = 27/xy \qquad (1)$$

The surface area S of the box is given by

$$S = 2(xy + xz + yz) = 2\left(xy + x \times \frac{27}{xy} + y \times \frac{27}{xy}\right) = 2\left(xy + \frac{27}{y} + \frac{27}{x}\right)$$

We need to find the minimum value of S. Therefore

$S_x = 2\left(y - \frac{27}{x^2}\right), S_y = 2\left(x - \frac{27}{y^2}\right), S_{xx} = 2\left(\frac{54}{x^3}\right) = \frac{108}{x^3}, S_{yy} = 2\left(\frac{54}{y^3}\right) = \frac{108}{y^3}, S_{xy} = 2.$

For critical points, we set

$S_x = 0 \Rightarrow 2\left(y - \frac{27}{x^2}\right) = 0 \Rightarrow y = \frac{27}{x^2}$ and $S_y = 0 \Rightarrow 2\left(x - \frac{27}{y^2}\right) = 0 \Rightarrow x = \frac{27}{y^2}$

Now, $x = \frac{27}{(27/x^2)^2} = 27 \times \frac{x^4}{27^2} = \frac{x^4}{27} \Rightarrow x - \frac{x^4}{27} = 0 \Rightarrow 27x - x^4 = 0$

$x(27 - x^3) = 0 \Rightarrow 27 - x^3 = 0, \quad x \neq 0$

➔ $x^3 = 27 \Rightarrow x = 3$ ➔ $y = 27/(3)^2 = 3$

Now, $D = (S_{xx})(S_{yy}) - (S_{xy})^2$

Therefore, (3, 3) $D = \left(\frac{108}{3^3}\right)\left(\frac{108}{3^3}\right) - (2)^2 = 12 > 0.$

Since $D > 0$ and $S_{xx} > 0$, therefore S has a minimum value at $x = y = 3$.

➔ $z = 27/(3 \times 3) = 3$

Hence, the dimensions of required box should be $3 \times 3 \times 3$ ft^3.

WORKSHEET 07

1. Verify the following using Euler's Theorem:

(i) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(ii) If $u = \sin^{-1}\frac{x+y}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

(iii) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

(iv) If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$

(v) If $u = \cos^{-1}\frac{x+y}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

(vi) If $u = \ln(x^2 + xy + y^2)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

2. The area of a triangle is given by $A = \frac{1}{2}ab\sin C$. When a = 20 cm, b = 30 cm and $\angle C$ = 30°, find

(a) The rate of change of A with respect to a, when b and C are constant.
(b) The rate of change of A with respect to b, when a and C are constant.
(c) The rate of change of A with respect to C, when a and b are constant.

3. Determine whether or not the following functions satisfy the Laplace equation

$$Z_{xx} + Z_{yy} = 0$$

(i) $z = e^x \cos y$ (ii) $z = \frac{1}{2}(e^{x+y})$ (iii) $z = x^2 - y^2$

4. In a certain electrostatic field, the potential is $u=\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-1/2}$ Find the rate of change of u in the positive x, y, and z directions, respectively, at the point (2, -2, -4).

5. If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R ohm resistor, the value of R can be found from the equation: $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$

Find the value of $\partial R/\partial R_3$, where $R_1=20, R_2=35$, and $R_3=50$.

6. A thin metal plate is being heated in such a way that the temperature T at any point (x, y) on the surface of the plate is given by: $T(x,y)=350-x^2-y^2$ degrees, where x and y are measured in centimeters. What is the temperature at the origin?

(a) On the basis of the function T, why will the temperature be greatest at the origin?
(b) Determine the rate of change of temperature with respect to distance at the point (10, 6) assuming x can vary and y is held constant?
(c) Determine the rate of change of temperature with respect to distance at the point (10, 6) assuming y can vary and x is held constant.
(d) Explain why the rates of change in parts (c) and (d) are negative.

7. Concern about body heat loss led to the development of a formula for measuring the surface area S of a person's body on the basis of the individual's weight w in kilograms and height h in centimeters: $S(w,h)=0.0072w^{0.425}h^{0.725}$

(a) Determine S(80, 178) (b) Find $S_w(w,h)$ and $S_h(w,h)$
(c) Find $S_w(w,h)$ and $S_h(w,h)$ at (80,178) (d) Explain the meaning of S_w and S_h

8. Find the approximate change in the hypotenuse of a right triangle of legs 6 and 8 inches when the shorter leg is extended by **1/4** inch and the longer leg is condensed by **1/8** inch.

9. The dimensions of a rectangular block of wood were found to be 10, 12, and 20 inches, with a possible error of 0.05 in each of the measurements. Find (approximately) the greatest error in the surface area of the block and the percentage error in the area caused by errors in the individual measurements.

10. Two sides of a triangle were measured as 150 ft and 200 ft, and the included angle is of $60°$. If the possible errors are 0.2ft in measuring the sides and $1° = \pi/180$ in the angle, what is the greatest possible error in the computed area?

11. The altitude of a right circular cone is 15 inches and is increasing at 0.2 in/min. The radius of the base is 10 inches and is decreasing at 0.3 in/min. How fast is the volume changing?

12. At a certain instant the radius of a right circular cylinder is 6 inches and is increasing at the rate 0.2 in/sec while the altitude is 8 inches and is decreasing at the rate 0.4 in/sec. Find the time rate of change (a) of the volume and (b) of the surface, at that instant.

13. Divide 120 into three non – negative parts such that the sum of their products taken two at a time is a maximum.

14. Show that a rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

15. A rectangular box, open at the top, is to have a volume of 32 cubic centimeters. Find the dimensions of the box requiring least material for its construction.

16. A manufacturer of aquarium wants to make a large rectangular box – shaped aquarium that will hold $64\,ft^3$ of water. If the material for the base costs \$20 per square foot and the material for the sides costs \$10 per square foot, find the dimensions for which the cost of the materials will be the least.

17. An open rectangular box is being made to contain a volume of 108 cubic feet. Find the dimensions that will minimize the amount of material used to make the box.

CHAPTER EIGHT

INDEFINITE INTEGRATION

8.1 INTRODUCTION

Readers are aware of the fact that there are two branches of calculus. They are **Differential Calculus** and **Integral Calculus.** In differential calculus we begin with a function f(x) and obtain its derivative $f'(x)$. Interpreting $f'(x)$ as a rate of change of f(x) led to a variety of applications. By contrast, there are situations in which we know the rate of change and seek the function f(x). We need to be able to reverse the differentiation process in such cases. In other words, in differential calculus we are given a function and we are required to find its derivative, while in integral calculus we are required to find the function whose derivative is given. This process is depicted as under:

Differentiation process

$$f(x) \longrightarrow f'(x) = F(x)$$

Anti derivative/Integration process

$$f(x) \longleftarrow f'(x) = F(x)$$

Thus integration is also known as anti-derivative. For example, if

Derivative of $(x^3) = 3x^2$ ➔ anti-derivative of $(3x^2) = x^3$.

Definition: If F(x) is a differentiable function such that $\frac{d}{dx}F(x) = f(x)$ then F(x) is called an integral or anti–derivative of f(x) and we write: $\int f(x)dx = F(x)$

Integration: The process of finding the integral of a function is called integration.

Integrand: The function to be integrated is called integrand. For example, if $\int f(x)dx = F(x)$ then f(x) is the integrand.

Integral sign: The symbol '$\int$' is called **integral sign** and is used to represent the process of integration. This symbol was first introduced by Leibniz.

Constant of Integration and Indefinite Integral

Let $\frac{d}{dx}F(x) = f(x) \Rightarrow \frac{d}{dx}[F(x)+c] = f(x)$, because $\frac{d}{dx}(c) = 0$.

Therefore, $\int f(x)dx = F(x)+c$

The arbitrary constant 'c' is called the constant of integration. It may be noted that $\int f(x)dx$ is called **indefinite integration.**

Properties of Indefinite Integrals

If f(x) and g(x) are any functions of the variable x, then following properties are always true.

(i) $\int[f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$ (ii) $\int k\,f(x)dx = k\int f(x)\,dx$

provided the anti-derivatives of f(x) and g(x) exist.

Table of Integrals of Elementary Functions

As discussed above the integration is reverse process of differentiation. We list now the integration of elementary functions showing both process of differentiation and anti-differentiation. It may be noted that after integration process is over we always add constant of integration.

1. $\frac{d}{dx}(x)=1 \quad \Rightarrow \int 1.dx = x+c$

2. If k is any constant then $\int k\,dx = k\int 1.dx = kx+c$. For example $\int 3\,dx = 3x+c$

3. $\frac{d}{dx}\frac{x^{n+1}}{n+1}=x^n \quad \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1}, n\neq -1$

4. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq 1$

5. $\frac{d}{dx}\ln x = \frac{1}{x} \quad \Rightarrow \int \frac{1}{x}dx = \ln x + c$

6. $\int \frac{1}{ax+b}dx = \frac{\ln(ax+b)}{a}+c$

7. $\frac{d}{dx}e^x = e^x \quad \Rightarrow \int e^x dx = e^x + c$

8. $\frac{d}{dx}e^{mx} = \frac{e^{mx}}{m} \Rightarrow \int e^{mx}dx = \frac{e^{mx}}{m}+c$

9. $\frac{d}{dx}a^x = a^x.\ln a \quad \Rightarrow \int a^x dx = \frac{a^x}{\ln a}+c$

10. $\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \int \cos x\,dx = \sin x + c$

11. $\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \int \sin x\,dx = -\cos x + c$

12. $\frac{d}{dx}\tan x = \sec^2 x \quad \Rightarrow \int \sec^2 x\,dx = \tan x + c$

13. $\frac{d}{dx}\cot x = -\text{cosec}^2 x \quad \Rightarrow \int \text{cosec}^2 x\,dx = -\cot x + c$

14. $\frac{d}{dx}\sec x = \sec x\tan x \quad \Rightarrow \int \sec x\tan x\,dx = \sec x + c$

15. $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x\cot x \quad \Rightarrow \int \text{cosec}\, x\cot x\,dx = -\text{cosec}\, x + c$

16. $\frac{d}{dx}\sin mx = m\cos mx \quad \Rightarrow \int \cos mx\,dx = \frac{\sin mx}{m}+c$

REMARK: **(i)** Observe that above formulae do not contain the integrals of tan x, cot x, sec x and cosec x. We shall discuss these in coming sections. **(ii)** In formula 13 observe the effect of multiple angle (mx) on the result. This is true for every formula from 7 to 12.

17. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c$

18. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow \int \frac{1}{\sqrt{1-x^2}}dx = \cos^{-1}x + c$

19. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \quad \Rightarrow \int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

20. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2} \quad \Rightarrow \int \frac{1}{1+x^2}dx = -\cot^{-1}x + c$

21. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \quad \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + c$

22. $\frac{d}{dx}\text{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}} \quad \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}dx = -\text{cosec}^{-1}x + c$

23. $\frac{d}{dx}\sinh x = \cosh x \quad \Rightarrow \int \cosh x\,dx = \sinh x + c$

24. $\frac{d}{dx}\cosh x = \sinh x \quad \Rightarrow \int \sinh x\,dx = \cosh x + c$

25. $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \quad \Rightarrow \int \operatorname{sech}^2 x\, dx = \tanh x + c$

26. $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x \quad \Rightarrow \int \operatorname{cosec}^2 x\, dx = -\coth x + c$

27. $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \quad \Rightarrow \int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + c$

28. $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x \quad \Rightarrow \int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + c$

29. $\frac{d}{dx}\sinh mx = m\cosh mx \quad \Rightarrow \int \cosh mx\, dx = \frac{\sinh mx}{m} + c$

30. $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}} \quad \Rightarrow \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c = \ln\left(x + \sqrt{x^2+1}\right)$

31. $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}} \quad \Rightarrow \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c = \ln\left(x + \sqrt{x^2-1}\right) + c$

32. $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} \quad \Rightarrow \int \frac{1}{1-x^2} dx = \tanh^{-1} x + c$

33. $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2} \quad \Rightarrow \int \frac{1}{1-x^2} dx = \coth^{-1} x + c$

34. $\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}} \quad \Rightarrow \int \frac{1}{x\sqrt{1-x^2}} dx = -\operatorname{sech}^{-1} x + c = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + c$

35. $\frac{d}{dx}\operatorname{csch}^{-1} x = \frac{-1}{x\sqrt{1+x^2}} \quad \Rightarrow \int \frac{1}{x\sqrt{1+x^2}} dx = -\operatorname{csch}^{-1} x + c = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) + c$

Example 01: Write down the anti-derivatives of the following functions.

(i) $f(x) = 0$: Since $\frac{d}{dx}(c) = 0 \Rightarrow \int 0.dx = c$

(ii) $f(x) = \sqrt{x}$: $\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

(iii) $f(x) = \frac{2x+3}{x}$: $\int \frac{2x+3}{x} dx = \frac{2}{3}\int x\, dx + 3\int \frac{1}{x} dx = \frac{2}{3}\frac{x^2}{2} + 3\ln x = \frac{x^2}{3} + 3\ln x + c$

(iv) $f(x) = \frac{x^2-3}{x^2+1}$:

$$\int \frac{x^2-3}{x^2+1} dx = \int \frac{x^2+1-4}{x^2+1} dx = \int \frac{x^2+1}{x^2+1} dx - 4\int \frac{1}{x^2+1} dx = \int 1 dx - 4\tan^{-1} x = x - 4\tan^{-1} x + c$$

(v) $f(x) = \tan^2 x$: $\int \tan^2 x\, dx = \int \left(\sec^2 x - 1\right) dx = \int \sec^2 x\, dx - \int 1 dx = \tan x - x + c$

(vi) $f(x) = \cot^2 x$: $\int \cot^2 x\, dx = \int \left(\operatorname{cosec}^2 x - 1\right) dx = \int \operatorname{cosec}^2 x\, dx - \int 1 dx = -\cot x - x + c$

(vii) $f(x) = \sin^2 x$:

$$\int \sin^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{2}\left(\int 1 dx - \int \cos 2x\, dx\right) = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

(viii) $f(x) = \cos^2 3x$:

$$\int \cos^2 3x\,dx = \int\left(\frac{1+\cos 6x}{2}\right)dx = \frac{1}{2}\left(\int 1dx + \int \cos 6x\,dx\right) = \frac{1}{2}\left(x + \frac{\sin 6x}{6}\right) + c$$

(ix) $f(x) = \sqrt{1-\cos x}$:

(x) $\int\sqrt{1-\cos x}\,dx = \int\sqrt{2\sin^2(x/2)}\,dx = \sqrt{2}\int\sin(x/2)dx = \sqrt{2}\left(-\frac{\cos(x/2)}{1/2}\right) = -2\sqrt{2}\cos\frac{x}{2} + c$

$f(x) = \sec^2 x\ \operatorname{cosec}^2 x$:

$$\int \sec^2 x \operatorname{cosec}^2 x dx = \int\frac{1}{\cos^2 x\sin^2 x}dx = 4\int\frac{1}{4\cos^2 x\sin^2 x}dx = 4\int\frac{1}{(2\cos x\sin x)^2}dx$$

$$= 4\int\frac{1}{(\sin 2x)^2}dx = 4\int \operatorname{cosec}^2 2x dx = -4\frac{\cot 2x}{2} + c = -2\cot 2x + c$$

8.2 METHODS OF INTEGRATION

There is no uniform technique to find integral of a given function. Several methods have been developed to evaluate the integration of various functions depending upon their nature. You will learn these methods gradually. To start with there are four major methods.

1. **Integration by substitution**
2. **Integration by parts**
3. **Integration of algebraic rational functions**
4. **Integration of algebraic irrational functions**
5. **Integration of rational trigonometric functions**

Integration by Substitution

This technique is used when we observe that integrand is the product or quotient of two functions where the derivative of one function is present in the integrand in some form.
Integration by substitution mostly involves the following two formulae:

- $\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c,\ n \neq -1$ **FORMULA-I** **[F-I]**
- $\int\frac{f'(x)}{f(x)}dx = \ln f(x) + c$ **FORMULA-II** **[F-II]**

We shall give reference of these formulae whenever we use them.

Example 01: Evaluate the following integrals

(i) $\int\left(ax^2+2bx+c\right)^n(ax+b)\,dx$

Solution: Here we observe that derivative of $(ax^2 + 2bx + c)$ is $2(ax + b)$ and $(ax + b)$ is present in multiplication form. So let,

$$z = ax^2 + 2bx + c \quad \Rightarrow dz = 2(ax+b)\,dx \quad \Rightarrow dz/2 = (ax+b)\,dx$$

Thus, $\int\left(ax^2+2bx+c\right)^n(ax+b)\,dx = \frac{1}{2}\int z^n dz = \frac{1}{2}\frac{z^{n+1}}{n+1} = \frac{\left(ax^2+2bx+c\right)^{n+1}}{2(n+1)} + C$ **[F-I]**

(ii) $\int\sqrt{1+\cos^2 x}\,\sin 2x\,dx$

Solution: Here we see that derivative of $1 + \cos^2 x$ is $-2\sin x\cos x = -\sin 2x$ is present in multiplication form, so let:

$$z = 1+\cos^2 x \quad \Rightarrow dz = -2\cos x\sin x\,dx = -\sin 2x\,dx \quad \Rightarrow -dz = \sin 2x\ dx$$

Thus, $\int\sqrt{1+\cos^2 x}\,\sin 2x\,dx = \int z^{1/2}(-dz) = -\frac{z^{3/2}}{3/2} = \frac{-2}{3}\left(1+\cos^2 x\right)^{3/2} + c$ **[F-I]**

(iii) $\int \frac{1}{x\ln x}dx$: Put $z = \ln x \Rightarrow \frac{dz}{dx} = \frac{1}{x} \Rightarrow dz = \frac{dx}{x}$

Now, $\int \frac{dx}{x\ln x} = \int \frac{1}{z}dz = \ln z + c = \ln(\ln x) + c$

(iv) $\int e^{\sin x}\cos x\, dx$: Put $z = \sin x \Rightarrow \frac{dz}{dx} = \cos x \Rightarrow dz = \cos x\, dx$

Thus $\int e^{\sin x}\cos x\, dx = \int e^z dz = e^z + c = e^{\sin x} + c$

(v) $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$: Put $z = \tan^{-1}x \Rightarrow \frac{dz}{dx} = \frac{1}{1+x^2} \Rightarrow dz = \frac{1}{1+x^2}dx$

Thus $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^z dz = e^z + c = e^{\tan^{-1}x} + c$

(vi) $\int (x^2+x)^4(2x+1)dx$: Put $z = (x^2+x) \Rightarrow \frac{dz}{dx} = 2x+1 \Rightarrow dz = (2x+1)dx$

Thus $\int (x^2+x)^4(2x+1)dx = \int z^4 dz = \frac{z^5}{5} + c = \frac{(x^2+x)^5}{5} + c$

(vii) $\int \frac{(2x+1)}{(x^2+x)}dx$: Put $z = (x^2+x) \Rightarrow \frac{dz}{dx} = 2x+1 \Rightarrow dz = (2x+1)dx$

Thus $\int \frac{(2x+1)}{(x^2+x)}dx = \int \frac{1}{z}dz = \ln z + c = \ln(x^2+x) + c$

(viii) $\int \tan x\, dx$

$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx = -\ln(\cos x) = \ln(\cos x)^{-1} = \ln\frac{1}{\cos x} = \ln\sec x + c$

(ix) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}dx = \ln\sin x + c$ **[F-II]**

(x) $\int \sec x\, dx = \int \sec x\frac{(\sec x + \tan x)}{(\sec x + \tan x)}dx = \int \frac{(\sec^2 x + \sec x\tan x)}{(\sec x + \tan x)}dx = \ln(\sec x + \tan x) + c$

(xi) $\int \csc x\, dx = \int \csc x\frac{(\csc x - \cot x)}{(\csc x - \cot x)}dx = \int \frac{(\csc^2 x - \csc x\cot x)}{(\csc x + \cot x)}dx = \ln(\csc x - \cot x) + c$

REMARK: (a) In parts (x) and (xi) we have used **[F-II]**

(b) $\frac{d}{dx}(\sec x + \tan x) = (\sec x\tan x + \sec^2 x)$ & $\frac{d}{dx}(\csc x - \cot x) = (-\csc x\cot x + \csc^2 x)$

(xii) $\int \sqrt{\frac{1+x}{1-x}}dx = \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx$

$= \int \frac{1}{\sqrt{1-x^2}}dx - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx = \sin^{-1}x - \frac{1}{2}(1-x^2)^{1/2} + c$ **[F-I]**

(xiii) $\int \frac{dx}{a+\sqrt{bx+c}}$: Put $\sqrt{bx+c} = z \Rightarrow bx + c = z^2 \Rightarrow b\,dx = 2z\,dz \Rightarrow dx = 2z\,dz/b$

$\Rightarrow I = \frac{2}{b}\int \frac{z}{a+z}dz = \frac{2}{b}\int \frac{z+a-a}{z+a}dz = \frac{2}{b}\left[\int \frac{z+a}{z+a}dz - a\int \frac{1}{z+a}dz\right] = \frac{2}{b}\left[\int 1\,dz - a\ln(z+a)\right]$

$$=\frac{2}{b}\left[z-a\ln(\sqrt{bx+c}+a)\right]+C=\frac{2}{b}\left[\sqrt{bx+c}-a\ln(\sqrt{bx+c}+a)\right]+C$$

(xiv) $\int\frac{dx}{\left(1+x^2\right)\tan^{-1}x}$: Put $z=\tan^{-1}x \Rightarrow dz=\frac{dx}{1+x^2}$

$$\Rightarrow I=\int\frac{1}{z}dz=\ln z+c=\ln\left(1+x^2\right)+c$$

(xv) $\int\frac{\sin x+\cos x}{\sin x-\cos x}dx$: Put $z=\sin x-\cos x \Rightarrow dz=(\cos x+\sin x)\ dx$

$$\Rightarrow I=\int\frac{1}{z}dz=\ln z+c=\ln\left(\sin x-\cos x\right)+c$$

(xvi) $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$: Put $z=\sqrt{x} \Rightarrow dz=\frac{1}{2}x^{-1/2}dx \Rightarrow 2dz=\frac{dx}{\sqrt{x}}$

$$\Rightarrow I=2\int\sin z\ dz=-2\cos z+c=-2\cos\sqrt{x}+c$$

(xvii) $\int\frac{dx}{e^x+e^{-x}}=\int\frac{dx}{\left(e^x+\frac{1}{e^x}\right)}=\int\frac{e^x}{\left(e^{2x}+1\right)}dx$: Put $z=e^x \Rightarrow dz=e^x\ dx$

$$\Rightarrow I=\int\frac{1}{z^2+1}dz=\tan^{-1}z+c=\tan^{-1}\left(e^x\right)+c$$

(xviii) $\int\frac{e^{2x}}{\sqrt{e^x-1}}dx$: Put $\sqrt{e^x-1}=z \Rightarrow e^x=1+z^2 \Rightarrow e^x dx=2z\ dz$

$$\Rightarrow I=\int\frac{e^x}{\sqrt{e^x-1}}(e^x dx)=\int\frac{1+z^2}{z}2z\ dz=\int\left(1+z^2\right)dz$$

$$=2\int\left(1+z^2\right)dz=2\left[z+\frac{z^3}{3}\right]+c=2\left[\sqrt{e^x-1}+\frac{\left(e^x-1\right)^{3/2}}{3}\right]+c$$

(xix) $\int\cos^7\theta\ \sin^3\theta\ d\theta=\int\cos^7\theta\ \sin^2\theta\sin\theta d\theta=\int\cos^7\theta\left(1-\cos^2\theta\right)\sin\theta d\theta$

$$=\int\cos^7\theta\ \sin\theta\ d\theta-\int\cos^9\theta\ \sin\theta\ d\theta=\frac{\cos^8\theta}{8}-\frac{\cos^{10}\theta}{10}+c \qquad \textbf{[F-I]}$$

(xx) $\int\sin^2\theta\ \cos^2\theta\ d\theta=\int\frac{(1-\cos 2\theta)}{2}.\frac{(1+\cos 2\theta)}{2}d\theta=\frac{1}{4}\int\left(1-\cos^2 2\theta\right)d\theta$

$$=\frac{1}{4}\int\left[1-\frac{(1+\cos 4\theta)}{2}\right]d\theta=\frac{1}{4}\int\frac{2-1-\cos 4\theta}{2}d\theta=\frac{1}{8}\int(1-\cos 4\theta)d\theta=\frac{1}{8}\left[\theta-\frac{\sin 4\theta}{4}\right]+c$$

(xxi) $\int\tan^3\theta\sec^3\theta\ d\theta=\int\tan^2\theta\ \sec^2\theta(\sec\theta\tan\theta)\ d\theta$

$$=\int\left(\sec^2\theta-1\right)\sec^2\theta(\sec\theta\tan\theta)\ d\theta=\int\left(\sec^4\theta-\sec^2\theta\right)(\sec\theta\tan\theta)\ d\theta$$

$$=\int\left(\sec^4\theta\right)(\sec\theta\tan\theta)\ d\theta+\int\left(\sec^2\theta\right)(\sec\theta\tan\theta)\ d\theta=\frac{\sec^5\theta}{5}+\frac{\sec^3\theta}{3}+c \qquad \textbf{[F-I]}$$

(xxii) $\int(2x+3)\sqrt{2x+1}\ dx$: Put $z=\sqrt{2x+1} \Rightarrow z^2=2x+1 \Rightarrow 2zdz=2dx \Rightarrow zdz=dx$

Also $x=(z^2-1)/2$. Thus given integral becomes:

$$I=\int\left(2.\frac{z^2-1}{2}+3\right)z.z\,dz=\int\left(z^2-1+3\right)z^2dz=\int\left(z^4+2z^2\right)dz=\frac{z^5}{5}+\frac{2z^3}{3}+c2$$

Put $z=\sqrt{2x+1}$, we get: $I=\frac{1}{5}(2x+1)^{5/2}+\frac{2}{3}(2x+1)^{3/2}+c$

(xxiii) $I=\int\frac{1}{\left(1+x^2\right)^{3/2}}dx$: Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

Thus, $I=\int\frac{1}{\left(1+\tan^2\theta\right)^{3/2}}\sec^2\theta d\theta=\int\frac{1}{\left(\sec^2\theta\right)^{3/2}}\sec^2\theta\, d\theta=\int\frac{\sec^2\theta}{\sec^3\theta}d\theta=\int\frac{1}{\sec\theta}d\theta$

$$=\int\cos\theta\, d\theta=\sin\theta+c \qquad (1)$$

Now, try to understand the following technique.

$\tan\theta=x=\frac{x}{1}=\frac{\text{Perp}}{\text{Base}}=\frac{P}{B} \Rightarrow P=x, B=1$. By Pythagorus Theorem:

$H=\sqrt{B^2+P^2}=\sqrt{1+x^2} \quad \Rightarrow \sin\theta=\frac{P}{H}=\frac{x}{\sqrt{1+x^2}}$

Thus, $\int\frac{1}{\left(1+x^2\right)^{3/2}}dx=\frac{x}{\sqrt{1+x^2}}+c$

(xxiv) $\int\frac{\tan x}{\cos x+\sec x}dx=\int\frac{\sin x}{\cos x\left(\cos x+\sec x\right)}dx=\int\frac{\sin x}{\cos^2 x+1}dx$ **[cos x sex x = 1]**

Putting z = cos x ➔ dz = - sinx dx ➔ -dz = sinx dx. Thus given integral becomes:

$$\int\frac{\tan x}{\cos x+\sec x}dx=-\int\frac{1}{z^2+1}dz=-\tan^{-1}z+c=\tan^{-1}(\cos x)+c$$

(xxv) $\int\frac{1}{\sin(x-a)\sin(x-b)}dx=\frac{1}{\sin(b-a)}\int\frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin(x-x+b-a)}{\sin(x-a)\sin(x-b)}dx=\frac{1}{\sin(b-a)}\int\frac{\sin[(x-a)-(x-b)]}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)]}{\sin(x-a)\sin(x-b)}dx$$

NOTE: sin(α - β) = sin α cos β - cos α sin β

$$=\frac{1}{\sin(b-a)}\left[\int\frac{\sin(x-a)\cos(x-b)}{\sin(x-a)\sin(x-b)}dx-\int\frac{\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx\right]$$

$$=\frac{1}{\sin(b-a)}\left[\int\frac{\cos(x-b)}{\sin(x-b)}dx-\int\frac{\cos(x-a)}{\sin(x-a)}dx\right]=\frac{1}{\sin(b-a)}\left\{\ln\left[\sin(x-b)\right]-\ln\left[\sin(x-a)\right]\right\}$$

(xxvi) $\int\frac{1}{(5\tan x+1)\cos^2 x}dx=\int\frac{\sec^2 x}{(5\tan x+1)}dx$: Putting $z=5\tan x+1 \Rightarrow dz=5\sec^2 x\,dx$

➔ dz/5 = $\sec^2 x$ dx. Thus given integration becomes:

$$\int\frac{1}{(5\tan x+1)\cos^2 x}dx=\frac{1}{5}\int\frac{1}{z}dz=\frac{1}{5}\ln z+c=\frac{1}{5}\ln\left(5\tan x+1\right)+c$$

(xxvii) $\int\sqrt{1+5\cos^2 x}\sin 2x\,dx$: Putting $z=1+5\cos^2 x \Rightarrow dz=-10\cos x\sin x\,dx$

➔ -dz/5 = 2sin x cos x dx = sin 2x dx. Thus given integral becomes:

$$\int\sqrt{1+5\cos^2 x}\,\sin 2x\,dx = -\frac{1}{5}\int\sqrt{z}\,dz = -\frac{1}{5}\frac{z^{3/2}}{3/2}+c = -\frac{2}{15}\left(1+5\cos^2 x\right)^{3/2}+c$$

(xxviii) $\int\frac{1}{2\sin^2 x+3\cos^2 x}dx = \int\frac{1}{2\cos^2 x\left(\tan^2 x+3/2\right)}dx = \frac{1}{2}\int\frac{\sec^2 x}{\left(\tan^2 x+3/2\right)}dx$

Putting z = tan x ➔ $dz = \sec^2 x\,dx$. Also let $3/2 = a^2$ ➔ $a = \sqrt{(3/2)}$. Thus

$$\int\frac{1}{2\sin^2 x+3\cos^2 x}dx = \frac{1}{2}\int\frac{1}{\left(z^2+a^2\right)}dz = \frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\frac{z}{a}+c$$

Putting the values of z and a, we obtain:

$$\int\frac{1}{2\sin^2 x+3\cos^2 x}dx = \frac{1}{2}\cdot\sqrt{\frac{2}{3}}\tan^{-1}\left(\tan\left[\sqrt{\frac{2}{3}}\,x\right]\right)+c = \sqrt{\frac{1}{6}}\tan^{-1}\left(\tan\left[\sqrt{\frac{2}{3}}\,x\right]\right)+c$$

(xxix) $\int\frac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}\,dx$: Putting $z=\sqrt{x} \Rightarrow dz = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\,dz = \frac{1}{\sqrt{x}}dx$

Thus given integral becomes:

$$\int\frac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}\,dx = 2\int\sec z\tan z\,dz = 2\sec z+c = 2\sec\sqrt{x}+c$$

(xxv) $\int\left(\pi^{\sin x}+[\sin x]^{\pi}\right)\cos x\,dx$: Putting z = sin x ➔ dz = cos x dx

Thus given integral becomes: $\int\left(\pi^{\sin x}+[\sin x]^{\pi}\right)\cos x\,dx$

$$=\int\left(\pi^{z}+z^{\pi}\right)dz = \int\pi^{z}dz+\int z^{\pi}dz = \frac{\pi^{z}}{\ln\pi}+\frac{z^{\pi+1}}{\pi+1}+c = \frac{\pi^{\sin x}}{\ln\pi}+\frac{[\sin x]^{\pi+1}}{\pi+1}+c$$

REMARK: We have used the formulae: $\int a^x dx = \frac{a^x}{\ln a}$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$

(xxvi) $\int\frac{\cos x}{3\sin x+4\sqrt{\sin x}}dx$: Putting z = sin x ➔ dz = cos x dx

Thus given integral becomes: $\int\frac{\cos x}{3\sin x+4\sqrt{\sin x}}dx = \int\frac{1}{3z+4\sqrt{z}}dz$ (1)

Substituting $u=\sqrt{z} \Rightarrow du = \frac{1}{2\sqrt{z}}dz \Rightarrow 2\sqrt{z}\,du = dz \Rightarrow 2u\,du = dz$

Thus (1) becomes:

$$\int\frac{\cos x}{3\sin x+4\sqrt{\sin x}}dx = \int\frac{1}{3u^2+4u}2u\,du = 2\int\frac{1}{u(3u+4)}u\,du = 2\int\frac{1}{(3u+4)}du$$

$$= 2\frac{\ln(3u+4)}{3}+c \qquad (2)$$

Put $u=\sqrt{z}=\sqrt{\sin x}$ in (2), we get

$$\int\frac{\cos x}{3\sin x+4\sqrt{\sin x}}dx = 2\frac{\ln\left(3\sqrt{\sin x}+4\right)}{3}+c$$

(xxv) $\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,(x\,dx)$

Putting $z^2 = x^2+1$ ➔ $x^2 = z^2-1$ ➔ 2x dx = 2 z dz ➔ x dx = z dz. Thus,

$$\int x^3\sqrt{x^2+1}\,dx=\int\left(z^2-1\right).z(z\,dz)=\int z^4dx-\int z^2dz=\frac{z^5}{5}-\frac{z^3}{3}+c$$

Now $z=\sqrt{x^2+1}$ ➔ $\int x^3\sqrt{x^2+1}\,dx=\dfrac{\left(x^2+1\right)^{5/2}}{5}-\dfrac{\left(x^2+1\right)^{3/2}}{3}+c$

Some well – known substitutions

The following substitutions are generally helpful to transform the integrand to an easier from.

If the integrand contains:	**Make the substitution:**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\sinh\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\cosh\theta$
$\sqrt{ax+b}$	$ax + b = z^2$

Example 02: Evaluate the following integrals

(i) $\displaystyle\int\frac{dx}{1+\sqrt{1+x}}$

Solution: Let $\sqrt{x+1}=z \Rightarrow x+1=z^2 \Rightarrow dx=2z\,dz$

$$\int\frac{dx}{1+\sqrt{1+x}}=\int\frac{2z}{1+z}dz=2\int\frac{1+z-1}{1+z}dz=2\left[\int 1dz-\int\frac{1}{1+z}dz\right]=2\left[z-\ln(1+z)\right]+c$$

$$=2\left[\sqrt{x+1}-\ln\left(1+\sqrt{x+1}\right)\right]+c$$

(ii) $\displaystyle\int\frac{dx}{\sqrt{x^2+4}}==\int\frac{dx}{\sqrt{4\left(x^2/4+1\right)}}=\int\frac{dx}{2\sqrt{(x/2)^2+1}}$

Put z = x/2 ➔ 2z = x ➔ 2dz = dx. Thus given integration becomes

$$I=\frac{2}{2}\int\frac{dz}{\sqrt{z^2+1}}=\sin^{-1}z+c=\sinh^{-1}\left(\frac{x}{2}\right)+c$$

(iii) $\displaystyle\int\frac{\sqrt{x^2-a^2}}{x^4}dx$: Put $x=a\sec\theta \Rightarrow dx=a\sec\theta\tan\theta\,d\theta$

$$I=\int\frac{\sqrt{a^2\sec^2\theta-a^2}}{(a\sec\theta)^4}a\sec\theta\tan\theta\,d\theta=\int\frac{a\tan\theta.a\sec\theta\tan\theta}{(a\sec\theta)^4}d\theta$$ **NOTE:** $\sec^2\theta-1=\tan^2\theta$

$$=\frac{1}{a^2}\int\frac{\tan^2\theta}{\sec^3\theta}d\theta=\frac{1}{a^2}\int\frac{\sin^2\theta}{\cos^2\theta.\sec^3\theta}d\theta=\frac{1}{a^2}\int\frac{\sin^2\theta.\cos^3\theta}{\cos^2\theta}d\theta=\frac{1}{a^2}\int(\sin\theta)^2.\cos\theta\,d\theta$$

$$=\frac{1}{a^2}\frac{\sin^3\theta}{3}+c \quad \textbf{[F-I]} \qquad (1)$$

Now, $x=a\sec\theta \Rightarrow \sec\theta=x/a \Rightarrow \cos\theta=a/x \Rightarrow \sin\theta=\sqrt{1-\cos^2\theta}$

$$\Rightarrow \sin\theta=\sqrt{1-\left(\frac{a}{x}\right)^2}=\sqrt{\frac{x^2-a^2}{x^2}}=\frac{\sqrt{x^2-a^2}}{x}$$

Substituting this in (1), we get: $I=\dfrac{1}{3a^2}\left[\dfrac{\sqrt{x^2-a^2}}{x}\right]^3+c=\dfrac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3}+c$

(iv) $\int \sqrt{a^2 - x^2}\,dx$: Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$.

Thus $I = \int \sqrt{a^2 - a^2\sin^2\theta}.a\cos\theta\, d\theta = \int a\sqrt{1-\sin^2\theta}.a\cos\theta\, d\theta = a^2\int \cos\theta.\cos\theta\, d\theta$

$$= a^2\int \cos^2\theta\, d\theta = a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c \qquad (1)$$

Now, $\sin\theta = x/a \;\rightarrow\; \theta = \sin^{-1}(x/a)$.

Also $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2\frac{x}{a}\sqrt{1-\left(\frac{x^2}{a^2}\right)} = \frac{2x\sqrt{a^2-x^2}}{a^2}$

Substituting these values in (1), we get:

$$I = \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{2x\sqrt{a^2-x^2}}{2a^2}\right] + c = \frac{1}{2}\left[\frac{1}{a}\sin^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2}\right] + c$$

Thus, $I = \int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + \frac{1}{a}\sin^{-1}\left(\frac{x}{a}\right)\right] + c$

Similarly, $I = \int\sqrt{a^2+x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2+x^2} + \frac{1}{a}\sinh^{-1}\left(\frac{x}{a}\right)\right] + c$

And, $I = \int\sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - \frac{1}{a}\cosh^{-1}\left(\frac{x}{a}\right)\right] + c$

REMARK: Students are advised to prove the last two results.

(v) $\int \frac{1}{x^2+a^2}dx$

Solution: Putting $x = a\tan\theta \;\rightarrow\; dx = a\sec^2\theta\, d\theta$

Thus, $\int\frac{1}{x^2+a^2}dx = \int\frac{1}{a^2\tan^2\theta + a^2}a\sec^2\theta\, d\theta = \frac{a}{a^2}\int\frac{1}{\tan^2\theta+1}\sec^2\theta\, d\theta$

$\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\int\frac{1}{\sec^2\theta}\sec^2\theta\, d\theta = \frac{1}{a}\int 1\, d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}x + c$

Integration by Parts

The product rule of differentiation leads to a method of integration called **integration by parts**. Consider, $\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx}$

Integrating both sides w.r.t x, we get: $uw = \int u\frac{dw}{dx}dx + \int w\frac{du}{dx}dx \qquad (1)$

Let $\frac{dw}{dx} = v \quad \Rightarrow w = \int v\,dx$. Substituting these in (1), we get:

$$u\int w\,dx = \int uv\,dx + \int\left(\int v\,dx\right)\frac{du}{dx}dx \;\rightarrow\; \int uv\,dx = u\int v\,dx = +\int\frac{du}{dx}\left(\int v\,dx\right)dx \qquad (2)$$

Formula (2) is known as integration by parts and is very much useful formula when one has to evaluate the integral of production of two functions.

Simple Tips to Select u and v

- **(i)** If x^n appears with sin mx or cos mx or e^{mx}, take $u = x^n$ and v the other functions.
- **(ii)** If x^n appears with ln x or any inverse trigonometric function, take $v = x^n$ and other function as u.
- **(iii)** If the integrand is of the form $e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$ then you are free

to choose any function as u and v.

(iv) If the integrand contains ln x only or any one of inverse trigonometric functions then multiply the integrand by 1 and consider v = 1 and u the given log function or inverse trigonometric function.

Example 03: Evaluate the following integrals using by parts formula

(i) $\int xe^x\,dx$: Consider u = x and v = e^x. Then

$$\int xe^x\,dx = x\int e^x\,dx - \int \frac{dx}{dx}\left(\int e^x dx\right)dx = xe^x - \int 1.e^x dx = xe^x - e^x + c = e^x(x-1)+c$$

(ii) $\int x^2 \ln x\,dx$: Consider u = ln x and v = x^2. Then

$$\int x^2\ln x\,dx = \ln x\int x^2\,dx - \int \frac{d}{dx}\ln x\left(\int x^2 dx\right)dx = \ln x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3\ln x}{3} - \frac{1}{3}\frac{x^3}{3} + c = \frac{x^3}{3}\left[\ln x - \frac{1}{3}\right]+c$$

(iii) $\int \ln x\,dx = \int \ln x.1dx$: Consider u = ln x and v = 1. Then

$$\int \ln x\,dx = \ln x\int 1dx - \int \frac{d}{dx}\ln x\left(\int 1dx\right)dx = x\ln x - \int \frac{1}{x}.x\ dx = x\ln x - \int 1dx$$

$$= x\ln x - x = x(\ln x - 1)+c$$

(iv) $\int x^2\tan^{-1}x\,dx$ **:** Consider u = $\tan^{-1}$ x and v = x^2. Then

$$\int x^2\tan^{-1}x\,dx = \tan^{-1}x\int x^2\,dx - \int \frac{d}{dx}\tan^{-1}x\left(\int x^2 dx\right)dx = \tan^{-1}x.\frac{x^3}{3} - \int \frac{1}{1+x^2}.\frac{x^3}{3}dx$$

$$= \frac{x^3\tan^{-1}x}{3} - \frac{1}{3}\int\frac{x^3}{1+x^2}dx \qquad (1)$$

Now consider, $\int \frac{x^3}{1+x^2}dx = \int \frac{x^2.x}{1+x^2}dx$: Putting $1 + x^2 = z$ ➔ $x^2 = z - 1$

➔ 2x dx = dz ➔ x dx = dz/2. Thus:

$$\int\frac{x^3}{1+x^2}dx = \int\frac{x^2.x}{1+x^2}dx = \int\frac{z-1}{z}dz = \int\left(\frac{z}{z}-\frac{1}{z}\right)dz = \int 1dz - \int\frac{1}{z}dz = z - \ln z + c$$

Putting the value of z = (1 + x^2) and using equation (1), we get:

$$\int x^2\tan^{-1}x\,dx = \frac{x^3\tan^{-1}x}{3} - \frac{1}{3}\left[\left(1+x^2\right) - \ln\left(1+x^2\right)\right]+c$$

(iv) $\int \sin^{-1}x\,dx = \int\sin^{-1}x(1)dx$ **:** Consider u = $\sin^{-1}$ x and v = 1. Then

$$\int\sin^{-1}x\,dx = \int\sin^{-1}x.1dx = \sin^{-1}x\int 1dx - \int\frac{d}{dx}\sin^{-1}x\left(\int 1dx\right)dx$$

$$= \sin^{-1}x.(x) - \int\frac{1}{\sqrt{1-x^2}}.(x)dx = x\sin^{-1}x + \frac{1}{2}\int\left(1-x^2\right)^{-1/2}(-2x)dx$$

$$\Rightarrow \int\sin^{-1}x\,dx = x\sin^{-1}x + \frac{1}{2}\frac{\left(1-x^2\right)^{1/2}}{1/2} + c = x\sin^{-1}x + \sqrt{1-x^2}+c$$

(v) $\int e^x\left(\frac{1+x\ln x}{x}\right)dx = \int e^x.\frac{1}{x}dx + \int\frac{x\ln x\,e^x}{x}dx = \int e^x.\frac{1}{x}dx + \int \ln x\,e^x\,dx \qquad (1)$

Now consider $\int e^x.\frac{1}{x}dx$: Let $u = e^x$ and $v = \ln x$. Using by parts formula, we get:

$$=\int e^x.\frac{1}{x}dx = e^x\int\frac{1}{x}dx - \int\frac{d}{dx}e^x\left(\int\frac{1}{x}dx\right)dx = e^x \ln x - \int e^x.\ln x\ dx$$

Thus equation (1) becomes:

$$\int e^x\left(\frac{1+x\ln x}{x}\right)dx = e^x \ln x - \int e^x \ln x\,dx + \int \ln x\, e^x\, dx = e^x \ln x + c$$

(vi) $\int x\sec^2 x\,dx$: Consider $u = x$ and $v = \sec^2 x$. Then

$$\int x\sec^2 x\,dx = x\int\sec^2 x\,dx - \int\frac{d}{dx}(x)\left(\int\sec^2 x\,dx\right)dx = x\tan x - \int 1.\tan x\,dx$$
$$= x\tan x - \ln\sec x + c$$

(vii) $\int\sec^3 x\,dx$

Let $I = \int\sec^3 x\,dx = \int\sec x.\sec^2 x\,dx$: Consider $u = \sec x$ and $v = \sec^2 x$. Then

$$I = \int\sec^3 x\,dx = \sec x\int\sec^2 x\,dx - \int\frac{d}{dx}(\sec x)\left(\int\sec^2 x\,dx\right)dx$$
$$= \sec x\tan x - \int\sec x\tan x.\tan x\,dx = \sec x\tan x - \int\sec x\tan^2 x\,dx$$
$$= \sec x\tan x - \int\sec x\left(\sec^2 x - 1\right)dx = \sec x\tan x - \int\sec^3 x\,dx - \int\sec x\,dx$$

$I = \sec x\tan x - I - \ln(\sec x + \tan x) \Rightarrow I + I = \sec x\tan x - \ln(\sec x + \tan x) = 2I$

Thus, $I = \int\sec^3 x\ dx = \frac{1}{2}\left[\sec x\tan x - \ln(\sec x + \tan x)\right] + c$

(viii) $\int\frac{x-\sin x}{1-\cos x}dx$: NOTE, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1-\cos x = 2\sin^2\frac{x}{2}$. Thus,

$$\int\frac{x-\sin x}{1-\cos x}dx = \int\frac{x}{2\sin^2 x/2}dx - \int\frac{2\sin x/2\cos x/2}{2\sin^2 x/2}dx = \frac{1}{2}\int x\csc^2\frac{x}{2}dx - \int\cot\frac{x}{2}dx$$

Consider the first integral: Taking $u = x$ and $v = \csc^2 x/2$ and integrating by parts:

$$\int\frac{x-\sin x}{1-\cos x}dx = \frac{1}{2}\left[x\left(-2\cot\frac{x}{2}\right)\right] - \int 1.\left(-2\cot\frac{x}{2}\right)dx + \int\cot\frac{x}{2}dx$$

$$\Rightarrow\int\frac{x-\sin x}{1-\cos x}dx = -x\cot\frac{x}{2} + \int\cot\frac{x}{2}dx - \int\cot\frac{x}{2}dx = -x\cot\frac{x}{2} + c$$

(ix) $\int e^x\frac{1-\sin x}{1-\cos x}dx$

$$\int e^x\frac{1-\sin x}{1-\cos x}dx = \int\frac{e^x}{2\sin^2 x/2}dx - \int e^x\frac{2\sin x/2\cos x/2}{2\sin^2 x/2}dx = \frac{1}{2}\int e^x\csc^2\frac{x}{2}dx - \int e^x\cot\frac{x}{2}dx$$

Consider the first integral: Taking $u = e^x$ and $v = \csc^2 x/2$ and integrating by parts:

$$\int e^x\frac{x-\sin x}{1-\cos x}dx = \frac{1}{2}\left[e^x\left(-2\cot\frac{x}{2}\right)\right] - \int e^x.\left(-2\cot\frac{x}{2}\right)dx + \int e^x\cot\frac{x}{2}dx$$

$$\Rightarrow\int\frac{x-\sin x}{1-\cos x}dx = -e^x\cot\frac{x}{2} + \int e^x\cot\frac{x}{2}dx - \int e^x\cot\frac{x}{2}dx = -e^x\cot\frac{x}{2} + c$$

(x) $\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\ dx$

Putting $x = \cos\theta$ ➔ $dx = -\sin\theta\,d\theta$. Moreover,

$\sqrt{\dfrac{1-x}{1+x}}=\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\dfrac{2\sin^2\theta/2}{2\cos^2\theta/2}}=\tan\left(\dfrac{\theta}{2}\right)$. Thus,

$$\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx=\int\tan^{-1}\left(\tan\frac{\theta}{2}\right).(-\sin\theta)\,d\theta=\int\frac{\theta}{2}(-\sin\theta)\,d\theta=-\frac{1}{2}\int\theta\sin\theta\,d\theta$$

Taking $u=\theta$ and $v=\sin\theta$ and integrating by parts, we get:

$$\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx=-\frac{1}{2}\left[\theta(-\cos\theta)-\int 1.(-\cos\theta)\,d\theta\right]=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]+c$$

Now $\cos\theta=x$ ➔ $\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-x^2}$. Also $\theta=\cos^{-1}x$. Thus

$$\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx=\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+c$$

(xi) $\int\sin^{-1}\sqrt{\dfrac{x}{x+a}}\,dx$

Putting $x=a\tan^2\theta$ ➔ $dx=2a\tan\theta\sec\theta\,d\theta$. Moreover,

$\sqrt{\dfrac{x}{x+a}}=\sqrt{\dfrac{a\tan^2\theta}{a\tan^2\theta+a}}=\sqrt{\dfrac{\tan^2\theta}{\sec^2\theta}}=\sqrt{\dfrac{\sin^2\theta}{\cos^2\theta.\sec^2\theta}}=\sqrt{\sin^2\theta}=\sin\theta$. Thus,

$$\int\sin^{-1}\sqrt{\frac{x}{x+a}}\,dx=\int\sin^{-1}(\sin\theta).(2a\tan\theta\sec\theta)\,d\theta=2a\int\theta.(\sec\theta\tan\theta)\,d\theta$$

Taking $u=\theta$ and $v=\sec\theta\tan\theta$ and integrating by parts, we get:

$$\int\sin^{-1}\sqrt{\frac{x}{x+a}}\,dx=2a\left[\theta.\sec\theta-\int 1.\sec\theta\,d\theta\right]=2a\left[\theta.\sec\theta-\ln(\sec\theta+\tan\theta)\right]+c \qquad (1)$$

Now $a\tan^2\theta=x \Rightarrow \tan\theta=\sqrt{\dfrac{x}{a}}$. $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\dfrac{x}{a}}=\sqrt{\dfrac{a+x}{a}}$ & $\theta=\tan^{-1}\sqrt{\dfrac{x}{a}}$

Thus, equation (1) becomes

$$\int\sin^{-1}\sqrt{\frac{x}{x+a}}\,dx=2a\left[\sqrt{\frac{a+x}{a}}\tan^{-1}\sqrt{\frac{x}{a}}-\ln\left(\frac{a}{\sqrt{a^2+x^2}}+\sqrt{\frac{x}{a}}\right)\right]+c$$

(xii) $\int e^{ax}\sin(bx+c)\,dx$: Integrating by parts taking $u=e^{ax}$ and $v=\sin(bx+c)$; we get

$$I=e^{ax}\left(\frac{-\cos(bx+c)}{b}\right)-\int a\,e^{ax}\left(\frac{-\cos(bx+c)}{b}\right)dx$$

$$=-e^{ax}\frac{\cos(bx+c)}{b}+\frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

Integrating by parts once again taking $u=e^{ax}$ and $v=\cos(bx+c)$

$$I=-e^{ax}\frac{\cos(bx+c)}{b}+\frac{a}{b}\left[e^{ax}\frac{\sin(bx+c)}{b}-\int ae^{ax}\frac{\sin(bx+c)}{b}dx\right]$$

$$=-e^{ax}\frac{\cos(bx+c)}{b}+\frac{a}{b^2}e^{ax}\sin(bx+c)-\frac{a^2}{b^2}\int e^{ax}\sin(bx+c)\,dx$$

$$I=\frac{a\sin(bx+c)-b\cos(bx+c)}{b^2}-\frac{a^2}{b^2}I \qquad \Rightarrow I+\frac{a^2}{b^2}I=\frac{a\sin(bx+c)-b\cos(bx+c)}{b^2}$$

$$\left(\frac{b^2+a^2}{b^2}\right)I=\frac{a\sin(bx+c)-b\cos(bx+c)}{b^2}$$

$$\Rightarrow I=\int e^{ax}\sin(bx+c)\,dx=\frac{1}{(a^2+b^2)}[a\sin(bx+c)-b\cos(bx+c)]+C$$

Similarly, $\int e^{ax}\cos(bx+c)\,dx=\frac{1}{(a^2+b^2)}[a\cos(bx+c)+b\sin(bx+c)]+C$ **[See Exercise]**

(xiii) $\int \ln\left(x+\sqrt{1+x^2}\right)dx=\int \ln\left(x+\sqrt{1+x^2}\right).1\,dx$

Take $v=1$ and $u=\ln\left(x+\sqrt{1+x^2}\right)$ $\Rightarrow \frac{du}{dx}=\frac{1}{x+\sqrt{1+x^2}}.\left[1+\frac{1}{2}\left(1+x^2\right)^{-1/2}.2x\right]$

$$\Rightarrow \frac{du}{dx}=\frac{1}{x+\sqrt{1+x^2}}.\left[1+\frac{x}{\sqrt{1+x^2}}\right]=\frac{1}{x+\sqrt{1+x^2}}.\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

Thus using integral by parts formula, we obtain:

$$\int \ln\left(x+\sqrt{1+x^2}\right)dx=\int \ln\left(x+\sqrt{1+x^2}\right).1\,dx=\ln\left(x+\sqrt{1+x^2}\right).x-\int\frac{1}{\sqrt{1+x^2}}.x\,dx$$

$$=\ln\left(x+\sqrt{1+x^2}\right).x-\frac{1}{2}\int\left(1+x^2\right)^{-1/2}(2x)\,dx=x\ln\left(x+\sqrt{1+x^2}\right)-\frac{1}{2}\frac{\left(1+x^2\right)^{1/2}}{1/2}+c$$

$$\Rightarrow \int \ln\left(x+\sqrt{1+x^2}\right)dx=x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+c.$$

(xiv) $\int\frac{x^2+1}{(x+1)^2}e^x\,dx=\int\frac{x^2+2x+1-2x}{(x+1)^2}e^x\,dx=\int\frac{(x+1)^2}{(x+1)^2}e^x\,dx-2\int\frac{x}{(x+1)^2}e^x\,dx$

$$=\int e^x\,dx-2\int\frac{x+1-1}{(x+1)^2}e^x\,dx=e^x-2\left[\int\frac{x+1}{(x+1)^2}e^x\,dx-\int\frac{1}{(x+1)^2}e^x\,dx\right]$$

$$\Rightarrow \int\frac{x^2+1}{(x+1)^2}e^x\,dx=e^x-2\left[\int\frac{1}{(x+1)}e^x\,dx-\int e^x.(x+1)^{-2}\,dx\right] \quad (1)$$

Consider the second integral and integrating by parts taking $u=e^x$ and $v=(x+1)^{-2}$

$$\int e^x.(x+1)^{-2}\,dx=e^x.(x+1)^{-1}-\int e^x(x+1)^{-1}\,dx=\frac{e^x}{(x+1)}-\int\frac{e^x}{(x+1)}dx$$

Thus equation (1) becomes:

$$\int\frac{x^2+1}{(x+1)^2}e^x\,dx=e^x-2\left[\int\frac{e^x}{(x+1)}dx-\frac{e^x}{(x+1)}-\int\frac{e^x}{(x+1)}dx\right]=e^x+2\frac{e^x}{(x+1)}+c$$

(xv) $\int\cos(\ln x)\,dx$: Putting $z=\ln x \Rightarrow x=e^z \Rightarrow dx=e^z dz$. Thus

$$\int\cos(\ln x)\,dx=\int e^z\cos z\,dz \quad (1)$$

Using the formula: $\int e^{ax}\cos(bx+c)\,dx=\frac{1}{(a^2+b^2)}[a\cos(bx+c)+b\sin(bx+c)]$

by taking $a=1$, $b=1$ and $c=0$, we get:

$$\int\cos(\ln x)\,dx=\int e^z\cos z\,dz=\frac{e^z}{1+1}[\cos z+\sin z]+c$$

Substituting $e^z=x$ and $z=\ln x$, we get:

$\int \cos(\ln x)\,dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + c$

(xvi) $\int \sqrt{x}\, e^{-\sqrt{x}} dx$: Putting $z = \sqrt{x} \Rightarrow dz = dx/2\sqrt{x} = dx/2z \Rightarrow dx = 2z\,dz$

Thus $\int \sqrt{x}\, e^{-\sqrt{x}} dx = -2\int z e^{-z} dz$. Integrating by parts, we get:

$$\int \sqrt{x}\, e^{-\sqrt{x}} dx = 2\int z e^{-z} dz = 2\left[-z e^{-z} - \int 1.\left(-e^{-z}\right) dz\right]$$

$$= 2\left[-z e^{-z} + \left(-e^{-z}\right)\right] + c = -2e^{-z}[z+1] + c$$

Substituting $z = \sqrt{x}$, we get: $\int \sqrt{x}\, e^{-\sqrt{x}} dx = -2e^{-\sqrt{x}}\left[\sqrt{x} + 1\right] + c$

(xvii) $I = \int x^5 e^{x^3} dx = \int x^3 e^{x^3} x^2 dx$: Putting $z = x^3 \Rightarrow dz = 3x^2 dx \Rightarrow dz/3 = x^2 dx$

Thus, $I = \int x^5 e^{x^3} dx = \frac{1}{3}\int z e^z dz = \frac{1}{3}\left[z e^z - \int 1.e^z dz\right] = \frac{1}{3}\left[z e^z - e^z\right] = \frac{e^z}{3}[z-1] + c$

Substituting $z = x^3$, we get : $I = \int x^5 e^{x^3} dx = \frac{e^{x^3}}{3}\left[x^3 - 1\right] + c$

Example 04: Show that $I = \int x^n \tan^{-1} x\, dx = \frac{x^{n+1}}{n+1}\tan^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2} dx$

Hence evaluate $I = \int x^3 \tan^{-1} x\, dx$. Also find the reduction formula for $I = \int x^n e^x dx$. Hence evaluate $I_4 = \int x^4 e^x dx$.

Solution: Let u= $\tan^{-1}x$ and v = x^n. Using by parts formula, we get:

$$I = \int x^n \tan^{-1} x\, dx = \tan^{-1} x . \frac{x^{n+1}}{n+1} - \int \frac{1}{1+x^2} . \frac{x^{n+1}}{1+n} dx = \frac{x^{n+1}}{n+1}\tan^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2} dx$$

Replacing n by 3, we have

$$\int x^3 \tan^{-1} x\, dx = \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \frac{x^4}{1+x^2} dx = \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx$$

$$I = \int x^3 \tan^{-1} x\, dx = \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x\right] + c$$

It may be noted that 'Reduction Formula is that formula where the power of x^n is reduced. It is also known as 'Recurrence Formula' means the formula that is repeatedly applied. Now consider:

$I = \int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - n\int x^{n-1} e^x dx$ NOTE : $u = x^n$, $v = e^x$

This is the reduction formula. Now,

$I_4 = \int x^4 e^x dx = x^4 e^x - \int 4x^3 e^x dx = x^4 e^x - 4\int x^3 e^x dx$. Apply this formula repeatedly, we get

$$I_4 = x^4 e^x - 4\left[x^3 e^x - 3\int x^2 e^x dx\right] = x^4 e^x - 4x^3 e^x + 12\int x^2 e^x dx$$

$$= x^4 e^x - 4x^3 e^x + 12\left[x^2 e^x - 2\int x\, e^x dx\right] = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\int x\, e^x dx$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left[x e^x - \int e^x dx\right] = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + c$$

Thus, $I_4 = \int x^4 e^x dx = e^x\left[x^4 - 4x^3 + 12x^2 - 24x + 24\right] + c$

Example 05: Evaluate of the following integrals and find their reduction formulae.

(i) $\int \sin^n x\,dx$ **(ii)** $\int \cos^n x\,dx$ **(iii)** $\int \tan^n x\,dx$

(iv) $\int \cot^n x\,dx$ **(v)** $\int \sec^n x\,dx$ **(vi)** $\int \csc^n x\,dx$

Solution: (i) Let $I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$. Taking $u = \sin^{n-1} x$ and $v = \sin x$ and integrating by parts, we get

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x(\cos x).(-\cos x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

➔ $I_n + (n-1)I_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$

➔ $(n - 1 + 1)\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$

$$I_n = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n} I_{n-2} \Rightarrow \int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

(ii) $\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx2$ **(Exercise)**

(iii) Let $I_n = \int \tan^n x\,dx = \int \tan^{n-2} x\tan^2 x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)dx$

$$= \int \tan^{n-2} x\sec^2 x\,dx - \int \tan^{n-2} x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

NOTE: We have used formula $\int (f(x))^n f'(x)dx = (f(x))^{n+1}/(n+1)$ in first integral

Thus $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \Rightarrow \int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(iv) $\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$ **(Exercise)**

(v) Let $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x\sec^2 x\,dx$. Taking $u = \sec^{n-2} x$ and $v = \sec^2 x$ and integrating by parts, we get

$$I_n = \sec^{n-2} x(\tan x) - \int (n-2)\sec^{n-3} x(\sec x\tan x).(\tan x)dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} x\tan^2 x\,dx = \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$I_n = \sec^{n-2} x\tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

➔ $I_n + (n-2)I_n = \sec^{n-2} x\tan x + (n-2)I_{n-2}$

➔ $(n - 2 + 1)\, I_n = \sec^{n-2} x \tan x + (n-2)\, I_{n-2}$

$$I_n = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

(vi) $I_n = \int \mathrm{cosec}^n x\,dx = -\frac{\mathrm{cosec}^{n-2} x\cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$ **[Exercise]**

Integration of Rational Algebraic Functions.

An expression of the form P(x)/Q(x) where P(x) and Q(x) are polynomials with real coefficients is called a rational function. In this section we shall study the integration of such functions by the methods of **partial fractions** and **completing the squares.**

In the fraction P(x)/Q(x), if the degree of the numerator is less than that of the denominator, the fraction is called a proper fraction, otherwise the improper fraction.

Integration by Partial Fractions

Before we study the integration by partial fractions, we must understand what partial fractions are? Consider the following expression.

$$\frac{1}{(x-1)}+\frac{2}{(x-3)}$$

If we simplify it, we get $\frac{1}{(x-1)}+\frac{2}{(x-3)}=\frac{(x-3)+2(x-1)}{(x-1)(x-3)}=\frac{3x-5}{(x-1)(x-3)}$

The expression, $\frac{3x-5}{(x-1)(x-3)}$ is called "Resultant Fraction" and the terms $1/(x-1)$ and $2/(x-3)$ are called its "Partial Fractions". Thus it is easy to find "Resultant Fraction" from "Partial Fractions". What about the reverse process? That is, is it possible to find the partial fractions from the resultant fraction? The answer is yes. But we have to learn the process to get partial fractions from the resultant fraction. The process is shown as under:

Let $\frac{3x-5}{(x-1)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-3)}$ (1)

➔ $\frac{3x-5}{(x-1)(x-3)}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}$ ➔ $3x-5=A(x-3)+B(x-1)$ (2)

Put $x-1=0$ ➔ $x=1$ in (2), we get: $3-5=A(1-3)$ ➔ $-2=-2A$ ➔ $A=1$

Put $x-3=0$ ➔ $x=3$ in (2), we get: $9-5=B(3-1)$ ➔ $4=2B$ ➔ $B=1$

Thus equation (1) becomes: $\frac{3x-5}{(x-1)(x-3)}=\frac{1}{(x-1)}+\frac{2}{(x-3)}$. This is same as above.

REMARK: It may be noted that before we resolve the resultant fraction into partial fractions, we must check that degree of a polynomial in the numerator is less than the degree of polynomial in the denominator. If yes, then we directly start resolving given expression into partial fraction technique. In case it isn't, we divide the numerator by the denominator till the degree of the polynomial in the numerator becomes less than degree of polynomial in the denominator. After this process is completed, we resolve the resultant fractions into partial fractions. We shall study THREE cases of partial fractions.

CASE 1: When all factors of the denominator D(x) are linear and distinct

For example, $\frac{2x-3}{(x-1)(x+1)(2x+3)}$

In this case, we write: $\frac{2x-3}{(x-1)(x+1)(2x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3}$

CASE 2: When the factors of D(x) are linear but some are repeated

For example, $\frac{x^2}{(x-1)^2(x+3)}$.

In this case, we write: $\frac{x^2}{(x-1)^2(x+3)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

CASE 3: When D(x) has non–repeated irreducible quadratic factors

For example, $x^2/[(x-1)(x^2+4)]$

In this case, we write: $\dfrac{x^2}{(x-1)(x^2+4)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+4)}$

All these cases are studied by means of examples. We shall directly consider the problems of evaluating the integrations which involve partial fractions.

Example 01: Evaluate the following integrals

(i) $\int \dfrac{x+1}{(x-2)(x-3)}dx$ (ii) $\int \dfrac{1}{(x^2-a^2)}dx$ (iii) $\int \dfrac{x^2}{(x-1)^3(x+1)}dx$

(vi) $\int \dfrac{x}{(x-1)(x^2+4)}dx$ (v) $\int \dfrac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx$

Solution: **(i)** Here degree of a polynomial in the numerator is less than the degree of polynomial in the denominator, so by using partial fractions, we have

$$\frac{x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad (1)$$

Multiplying both sides of (1) by (x – 2) (x – 3), we get

$$x + 1 = A(x-3) + B(x-2) \quad (2)$$

Now put $x - 2 = 0 \Rightarrow x = 2$ in (2), we have

$$2 + 1 = A(2-3) \Rightarrow A = -3$$

Again put $x - 3 = 0 \Rightarrow x = 3$ into (2), we have

$$3 + 1 = B(3-2) \Rightarrow B = 4$$

Thus (1) becomes: $\dfrac{x+1}{(x-2)(x-3)} = \dfrac{-3}{x-2} + \dfrac{4}{x-3}$ (3)

Now integrating, we have

$$\int \frac{x+1}{(x-2)(x-3)}dx = -3\int \frac{1}{x-2}dx + 4\int \frac{1}{x-3}dx = -3\ln(x-2) + 4\ln(x-3) + c$$

(ii) $\dfrac{1}{(x^2-a^2)} = \dfrac{1}{(x-a)(x+a)} = \dfrac{A}{x-a} + \dfrac{B}{x+a}$ (1)

Multiplying both sides of (1) by (x – a) (x + a), we get

$$1 = A(x+a) + B(x-a) \quad (2)$$

Now put $x - a = 0 \Rightarrow x = a$ in (2), we have

$$1 = A(a+a) \Rightarrow A = 1/2a$$

Again put $x + a = 0 \Rightarrow x = -a$ into (2), we have: $1 = B(-a-a) \Rightarrow B = -1/2a$

Thus (1) becomes: $\dfrac{1}{(x^2-a^2)} = \dfrac{1}{(x-a)(x+a)} = \dfrac{1/2a}{x-a} + \dfrac{-1/2a}{x+a}$ (3)

Now integrating, we have

$$\int \frac{1}{(x^2-a^2)}dx = \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right] = \frac{1}{2a}\left[\ln(x-a) - \ln(x+a)\right] + c$$

$$\Rightarrow \int \frac{1}{(x^2-a^2)}dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$$

Similarly, $\int \dfrac{1}{(a^2-x^2)}dx = \dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right) + c$ **[Left as Exercise]**

(iii) $\int \dfrac{x^2}{(x-1)^3(x+1)}dx$

Solution: Consider $\dfrac{x^2}{(x-1)^3(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{x+1}$ (1)

Multiplying both sides by $(x-1)^3(x+1)$, we get

$$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \quad (2)$$

Substituting $x - 1 = 0 \Rightarrow x = 1$ in (2), we get: $1 = 2C \Rightarrow C = 1/2$

Substituting $x + 1 = 0 \Rightarrow x = -1$ in (2), we get: $1 = -8D \Rightarrow D = -1/8$

You may observe that in (1), there are four unknowns A, B, C and D but there are only two factors $(x-1)$ and $(x+1)$. Thus only two unknown C and D are found

To find the remaining unknowns A and B, we rewrite the equation (2) in simplified form:

$$x^2 = A(x^3 - x^2 - x + 1) + B((x^2 - 1) + C(x + 1) + D(x^3 - 3x^2 + 3x - 1)$$

Comparing the coefficients of:

x^3: $0 = A + D \Rightarrow A = -D \Rightarrow A = 1/8$ [Since D = -1/8]

x^2: $1 = -A + B - 3D \Rightarrow B = 1 + A + 3D = 1 + 1/8 - 3/8 = 6/8 = 3/4$

Thus $A = 1/8$, $B = 3/4$, $C = 1/2$ and $D = -1/8$

Substituting all these values into equation (1), we get

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} - \frac{1}{8(x+1)}$$

Integrating both sides with respect to x, we have

$$\int \frac{x^2}{(x-1)^3(x+1)}dx = \frac{1}{8}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{dx}{(x-1)^3} - \frac{1}{8}\int\frac{dx}{x+1}$$

$$= \frac{1}{8}\int\frac{dx}{x-1} + \frac{3}{4}\int(x-1)^{-2}dx + \frac{1}{2}\int(x-1)^{-3}dx - \frac{1}{8}\int\frac{dx}{x+1}$$

$$= \frac{1}{8}\ln(x-1) + \frac{3}{4}\frac{(x-1)^{-2+1}}{(-2+1)} + \frac{1}{2}\frac{(x-1)^{-3+1}}{(-3+1)} - \frac{1}{8}\ln(x+1) + c$$

Thus, $\displaystyle\int \frac{x^2}{(x-1)^3(x+1)}dx = \frac{1}{8}\ln(x-1) - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{8}\ln(x+1) + c$

(iii) $\displaystyle\int \frac{x}{(x-1)(x^2+4)}dx$: Consider $\dfrac{x}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{(x^2+4)}$ (1)

Multiplying (1) by $(x-1)(x^2+4)$, we get

$x = A(x^2 + 4)(Bx + C)(x - 1)$ (2)

Substituting $x - 1 = 0 \Rightarrow x = 1$ in (2) we have: $1 = A(1 + 4) \Rightarrow A = 1/5$

Now to find B and C, we simplify (2) to get;

$$x = Ax^2 + 4A + Bx^2 - Bx + Cx - C \quad (3)$$

Now comparing the coefficients of:

x^2: $0 = A + B \Rightarrow B = -A \Rightarrow B = -1/5$

x: $1 = -B + C \Rightarrow C = 1 + B = 1 - 1/5 \Rightarrow C = 4/5$

Substituting the values of A, B and C in (1), we have

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} + \frac{-x+4}{5(x^2+4)} = \frac{1}{5(x-1)} - \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)}$$

Now integrating, we have

$$\int\frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int\frac{1}{x-1}dx - \frac{1}{5}\int\frac{x}{(x^2+4)}dx + \frac{4}{5}\int\frac{1}{(x^2+4)}dx$$

$$=\frac{1}{5}\int\frac{1}{x-1}dx-\frac{1}{10}\int\frac{2x}{(x^2+4)}dx+\frac{4}{5}\int\frac{1}{(x^2+4)}dx$$

$$=\frac{1}{5}\ln(x-1)-\frac{1}{10}\ln(x^2+4)+\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Thus, $\int\frac{x}{(x-1)(x^2+4)}dx=\frac{1}{5}\ln(x-1)-\frac{1}{10}\ln(x^2+4)+\frac{2}{5}\tan^{-1}\frac{x}{2}+c$

(iv) $\int\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx$

Let $z=\sin x$ ➔ $dz=\cos x\,dx$. Therefore,

$$I=\int\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx=\int\frac{1}{(1+z)(2+z)(3+z)}dz$$

Now, $\frac{1}{(1+z)(2+z)(3+z)}=\frac{A}{1+z}+\frac{B}{2+z}+\frac{C}{3+z}$ (1)

Multiplying (1) by $(1+z)(2+z)(3+z)$ to get

$1=A(2+z)(3+z)+B(1+z)(3+z)+C(1+z)(2+z)$ (2)

Substituting $1+z=0$ ➔ $z=-1$ into (2), we get

$1=A(2-1)(3-1)=2A$ ➔ $A=1/2$

Substituting $2+z=0$ ➔ $z=-2$ into (2) to get

$1=B(1-2)(3-2)=-B$ ➔ $B=-1$

Substituting $3+z=0$ ➔ $z=-3$ into (2) to get

$1=C((1-3)(2-3)=2C$ ➔ $C=1/2$

Hence, equation (1) becomes: $\frac{1}{(1+z)(2+z)(3+z)}=\frac{1/2}{1+z}-\frac{1}{2+z}+\frac{1/2}{3+z}$

Integrating, we get

$$\int\frac{1}{(1+z)(2+z)(3+z)}dz=\frac{1}{2}\int\frac{1}{1+z}dz-\int\frac{1}{2+z}dz+\frac{1}{2}\int\frac{1}{3+z}dz$$

$$=\frac{1}{2}\ln(1+z)-\ln(2+z)+\frac{1}{2}\ln(3+z)+c$$

Replacing z by sin x, we get

$$\int\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx=\frac{1}{2}\ln(1+\sin x)-\ln(2+\sin x)+\frac{1}{2}\ln(3+\sin x)+c$$

Integration by Completing the Squares Method

If the rational algebraic function P(x)/Q(x) is such that Q(x) is quadratic function that do not have real factors, the method of completing the squares is used to evaluate the integrations of P(x)/Q(x). The method is well presented by the following two examples.

Example 02: Evaluate the following by "Completing the Squares Method".

(i) $\int\frac{2}{x^2+4x+5}dx$ **(ii)** $\int\frac{x-1}{x^2+4x+5}dx$ **(iii)** $\int\frac{x^2+x+1}{x^2+4x+5}dx$

Solution: (i) Consider $\int\frac{1}{x^2+4x+5}dx=\int\frac{1}{(x^2+4x+2^2)-2^2+5}dx=\int\frac{1}{(x+2)^2+1}dx$

Putting: $z=x+2$ ➔ $dz=dx$. Thus

$$\int\frac{1}{x^2+4x+5}dx=\int\frac{1}{z^2+1}dz=\tan^{-1}z+c=\tan^{-1}(x+2)+c$$

(ii) Consider, $\int \frac{x-1}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2x-2}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2x+4-4-2}{x^2+4x+5}dx$. Thus,

$$\int \frac{x-1}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}dx - \frac{6}{2}\int \frac{1}{x^2+4x+5}dx = \frac{1}{2}\ln(x^2+4x+5) - 3I_1 \quad (1)$$

Here, $I_1 = \int \frac{1}{x^2+4x+5}dx = \tan^{-1}(x+2)$. [See part (i)]. Thus equation (1) becomes:

$$\int \frac{x-1}{x^2+4x+5}dx = \frac{1}{2}\ln(x^2+4x+5) - 3\tan^{-1}(x+2) + c$$

(iii) $\int \frac{x^2+x+1}{x^2+4x+5}dx = \int \frac{(x^2+4x+5)-4x-5+x+1}{x^2+4x+5}dx$

$$= \int \frac{x^2+4x+5}{x^2+4x+5}dx + \int \frac{-3x-4}{x^2+4x+5}dx = \int 1dx - 3\int \frac{x+(4/3)}{x^2+4x+5}dx$$

$$= x - \frac{3}{2}\int \frac{2x+(8/3)}{x^2+4x+5}dx = x - \frac{3}{2}\int \frac{2x+4-4+(8/3)}{x^2+4x+5}dx$$

$$= x - \frac{3}{2}\int \frac{2x+4}{x^2+4x+5}dx - \frac{3}{2}\int \frac{-4+(8/3)}{x^2+4x+5}dx = x - \frac{3}{2}\ln(x^2+4x+5) - \frac{3}{2}\int \frac{-4/3}{x^2+4x+5}dx$$

$$= x - \frac{3}{2}\ln(x^2+4x+5) - 2\int \frac{1}{x^2+4x+5}dx \quad (1)$$

From part (i) $\int \frac{1}{x^2+4x+5}dx = \tan^{-1}(x+2)$. Thus (1) becomes:

$$\int \frac{x^2+x+1}{x^2+4x+5}dx = x - \frac{3}{2}\ln(x^2+4x+5) - 2\tan^{-1}(x+2) + c$$

Example 03: Evaluate $\int \frac{1}{x^4+1}dx$

Solution: Consider

$$I = \int \frac{1}{x^4+1}dx = \frac{1}{2}\int \frac{2}{x^4+1}dx = \frac{1}{2}\int \frac{x^2+1-x^2+1}{x^4+1}dx = \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx \quad (1)$$

Now consider, $I_1 = \int \frac{x^2+1}{x^4+1}dx = \int \frac{x^2(1+1/x^2)}{x^2(x^2+1/x^2)}dx = \int \frac{(1+1/x^2)}{(x^2+1/x^2)}dx$

Let $z = x - 1/x$ ➔ $dz = (1 + 1/x^2)\,dx$. Also $z^2 = x^2 - 2 + 1/x^2$ ➔ $x^2 + 1/x^2 = z^2 + 2$. Thus,

$$I_1 = \int \frac{1}{z^2+2}dz = \int \frac{1}{z^2+\left(\sqrt{2}\right)^2}dz = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(x-1/x)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{(x^2-1)}{\sqrt{2}x}$$

Now consider, $I_2 = \int \frac{x^2-1}{x^4+1}dx = \int \frac{x^2(1-1/x^2)}{x^2(x^2+1/x^2)}dx = \int \frac{(1-1/x^2)}{(x^2+1/x^2)}dx$

Let $z = x + 1/x$ ➔ $dz = (1 - 1/x^2)\,dx$. Also $z^2 = x^2 + 2 + 1/x^2$ ➔ $x^2 + 1/x^2 = z^2 - 2$. Thus,

$$I_2 = \int \frac{1}{z^2-2}dz = \int \frac{1}{z^2-\left(\sqrt{2}\right)^2}dz = \frac{1}{2\sqrt{2}}\ln\frac{z-\sqrt{2}}{z+\sqrt{2}} = \frac{1}{2\sqrt{2}}\ln\frac{(x-1/x-\sqrt{2})}{(x-1/x+\sqrt{2})}$$

$$= \frac{1}{2\sqrt{2}}\ln\frac{(x^2-\sqrt{2}x-1)}{(x^2+\sqrt{2}x-1)}$$

Thus equation (1) becomes:

$$\int \frac{1}{x^4+1}dx = \frac{1}{2}[I_1 - I_2] = \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x^2-1)}{\sqrt{2}x} - \frac{1}{2\sqrt{2}}\ln\frac{(x^2-\sqrt{2}x-1)}{(x^2+\sqrt{2}x-1)}\right]+c$$

REMARK: The above example does not belong to "Completing the Squares Method", nevertheless it is an important type of integral. Students are advised to see the steps taken in solving this integral.

The following formulae have been used here.

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\frac{x-a}{x+a} \text{ and } \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

Integration of Irrational Algebraic Functions

In this section, we shall study the integrals of form $\sqrt{P(x)/Q(x)}$ where P(x) and Q(x) are polynomials with $Q(x) \neq 0$. The integrand may contain one of the functions $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$. In such cases, we know that following substitutions are made.

If the integrand contains:	$\sqrt{a^2-x^2}$	Substitute	$x = a\sin\theta$
	$\sqrt{a^2+x^2}$	Substitute	$x = a\tan\theta$ or $x = a\sinh\theta$
	$\sqrt{x^2-a^2}$	Substitute	$x = a\sec\theta$ or $x = a\cosh\theta$

Example 01: Evaluate the following integrals

(i) $\int \frac{1}{\sqrt{x^2+4x+13}}dx$ (ii) $\int \frac{x+3}{\sqrt{x^2+4x+5}}dx$ (iii) $\int x^2\sqrt{25-x^2}\,dx$

(iv) $\int e^x\sqrt{1-e^{2x}}\,dx$ (v) $\int \sqrt{x^2+4x+5}\,dx$ (vi) $\int \frac{x^2-3}{x\sqrt{x^2+4}}dx$

Solution: (i) $\int \frac{1}{\sqrt{x^2+4x+13}}dx = \int \frac{1}{\sqrt{(x^2+4x+4)+9}}dx = \int \frac{1}{\sqrt{(x+2)^2+3^2}}dx$

Let $x + 2 = z$ ➔ $dx = dz$. Thus,

$$\int \frac{1}{\sqrt{x^2+4x+13}}dx = \int \frac{1}{\sqrt{z^2+3^2}}dz = \sinh^{-1}\left(\frac{z}{3}\right)+c = \sinh^{-1}\left(\frac{x+2}{3}\right)+c$$

(ii) $\int \frac{x+3}{\sqrt{x^2+4x+5}}dx = \frac{1}{2}\int \frac{2x+6}{\sqrt{x^2+4x+5}}dx = \frac{1}{2}\int \frac{2x+4+2}{\sqrt{x^2+4x+5}}dx$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+4x+5}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+4x+5}}dx$$

$$= \frac{1}{2}\int \left(x^2+4x+5\right)^{-1/2}(2x+4)dx + \int \frac{1}{\sqrt{(x^2+4x+4)+1}}dx \text{. Thus}$$

$$\int \frac{x+3}{\sqrt{x^2+4x+5}}dx = \frac{1}{2}\frac{\left(x^2+4x+5\right)^{1/2}}{1/2} + \int \frac{1}{\sqrt{(x+2)^2+1}}dx$$

$$= \sqrt{x^2+4x+5} + \sinh^{-1}(x+2) + c \quad \text{[see part (i)]}$$

(iii) Let $I = \int x^2\sqrt{25-x^2}\,dx$ Put $x = 5\sin\theta$ ➔ $dx = a\cos\theta\,d\theta$. Then

$$I = \int x^2\sqrt{25-x^2}dx = \int 25\sin^2\theta\sqrt{25-25\sin^2\theta}\,(5\cos\theta)\,d\theta$$

$$= 625\int \sin^2\theta\cos\theta\sqrt{1-\sin^2\theta}\ d\theta = 625\int \sin^2\theta\cos^2\theta\ d\theta$$

$$= 625\int\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{625}{4}\int\left(1-\cos^2 2\theta\right)d\theta$$

$$= \frac{625}{4}\left[\int 1\,d\theta - \int\left(\frac{1+\cos 4\theta}{2}\right)d\theta\right] = \frac{625}{4}\left(\theta - \frac{1}{2}\int 1\,d\theta - \frac{1}{2}\int \cos 4\theta\ d\theta\right)$$

$$= \frac{625}{4}\left(\theta - \frac{\theta}{2} - \frac{\sin 4\theta}{8}\right) = \frac{625}{4}\left(\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right) = \frac{625}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)$$

Replacing θ by $\sin^{-1}(x/5)$, we get

$$I = \int x^2\sqrt{25-x^2}\ dx = \frac{625}{8}\left[\sin^{-1}\left(\frac{x}{5}\right) - \frac{1}{4}\sin 4\left(\sin^{-1}\frac{x}{5}\right)\right] + c$$

(iv) $\int e^x\sqrt{1-e^{2x}}\ dx$: Putting $z = e^x$ ➔ $dx = e^x\,dx$. Then

$$I = \int e^x\sqrt{1-e^{2x}}\,dx = \int z\sqrt{1-z^2}\left(\frac{dz}{z}\right) = \int\sqrt{1-z^2}\,dz$$

$$I = \frac{z}{2}\sqrt{1-z^2} + \frac{1}{2}\sin^{-1} z \qquad \left[\text{using } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

Replacing z by e^x, we get

$$I = \int e^x\sqrt{1-e^{2x}}\ dx = \frac{e^x}{2}\sqrt{1-e^{2x}} + \frac{1}{2}\sin^{-1}\left(e^x\right) + c$$

(v) $\int\sqrt{x^2+4x+5}\ dx = \int\sqrt{(x^2+4x+4)+1}\ dx = \int\sqrt{(x+2)^2+1}\ dx$

Let $x + 2 = z$ ➔ $dx = dz$. Thus,

$$\int\sqrt{x^2+4x+5}\ dx = \int\sqrt{z^2+1}\ dz = \frac{z\sqrt{z^2+1}}{2} + \frac{1}{2}\sinh^{-1} z + c \qquad \textbf{[Note this formula]}$$

Substituting $z = x + 2$, we get:

$$\int\sqrt{x^2+4x+5}\ dx = \frac{(x+2)\sqrt{(x+2)^2+1}}{2} + \frac{1}{2}\sinh^{-1}(x+2) + c$$

$$➔ \int\sqrt{x^2+4x+5}\ dx = \frac{(x+2)\sqrt{x^2+4x+5}}{2} + \frac{1}{2}\sinh^{-1}(x+2) + c$$

(vi) $\int\frac{x^2-3}{x\sqrt{x^2+4}}dx$: Putting $x = 2\tan\theta$ ➔ $dx = 2\sec^2\theta\ d\theta$. Thus

$$\int\frac{x^2-3}{x\sqrt{x^2+4}}dx = \int\frac{2\tan^2\theta-3}{2\tan\theta\sqrt{\tan^2\theta+4}}2\sec^2\theta\ d\theta = \int\frac{2\tan^2\theta-3}{2\tan\theta\ 2\sec^2\theta}2\sec^2\theta\ d\theta$$

$$= \int\frac{2\tan^2\theta-3}{2\tan\theta}\ d\theta = \int\frac{2\tan^2\theta}{2\tan\theta}\ d\theta - 3\int\frac{1}{2\tan\theta}\ d\theta = \int\tan\theta\ d\theta - \frac{3}{2}\int\cot\theta\ d\theta$$

$$= \ln\sec\theta - \frac{3}{2}\ln(\csc\theta - \cot\theta) + c \qquad (1)$$

Now, $x = 2\tan\theta$ ➔ $\tan\theta = x/2$ ➔ $\cot\theta = 2/x$. Also

$$\csc\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\left(\frac{2}{x}\right)^2} = \frac{\sqrt{x^2+4}}{x},\quad \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{2}\right)^2} = \frac{\sqrt{4+x^2}}{2}$$

Thus (1) becomes:

$$\int \frac{x^2-3}{x\sqrt{x^2+4}}dx = \ln\left(\frac{\sqrt{4+x^2}}{2}\right) - \frac{3}{2}\ln\left[\frac{\sqrt{x^2+4}}{x} - \frac{2}{x}\right] + c$$

Four Standard Cases

In this section, we shall study four important forms of integrations.

FORM-I: Integral of the form $\int \frac{1}{\text{Linear factor}\sqrt{\text{Linear factor}}}dx$

Rule to evaluate: Put $\sqrt{\text{Linear factor}} = z$

FORM-II: Integral of the form $\int \frac{1}{\text{Quadratic factor}\sqrt{\text{Linear factor}}}dx$

Rule to evaluate: Put $\sqrt{\text{Linear factor}} = z$

FORM-III: Integral of the form $\int \frac{1}{\text{Linear factor}\sqrt{\text{Quadratic factor}}}dx$

Rule to evaluate: Put Linear factor = 1/z

CASE 4: Integral of the form $\int \frac{1}{\text{Quadratic factor}\sqrt{\text{Quadratic factor}}}dx$

Rule to evaluate: Put x = 1/y and in the resulting integral put $\sqrt{\text{Quadratic factor}} = z$

Example 02: Evaluate the following integrals

(i) $\int \frac{1}{(2x+3)\sqrt{x+5}}dx$ (ii) $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx$

(iii) $\int \frac{1}{(x^2-2x+2)\sqrt{x-1}}dx$ (iv) $\int \frac{1}{x^2\sqrt{x^2+1}}dx$

Solution: (i) Let $I = \int \frac{1}{(2x+3)\sqrt{x+5}}dx$ $\left[\text{form} \int \frac{1dx}{\text{Linear}\sqrt{\text{Linear}}}\right]$

Put $z=\sqrt{x+5} \Rightarrow z^2 = x+5 \Rightarrow 2z\,dz = dx$. Then

$$I = \int \frac{2zdz}{\left[2(z^2-5)+3\right](z)} = \int \frac{2dz}{2z^2-10+3} = \frac{2}{2}\int \frac{dz}{z^2-7/2} = \int \frac{dz}{z^2-\left(\sqrt{7/2}\right)^2}$$

$$I = \frac{1}{2\sqrt{7/2}}\ln\frac{z-\sqrt{7/2}}{z+\sqrt{7/2}} = \frac{1}{\sqrt{14}}\ln\frac{\sqrt{x+5}-\sqrt{7/2}}{\sqrt{x+5}+\sqrt{7/2}} + c$$

(ii) Let $I = \int \frac{x\,dx}{(x^2-2x+2)\sqrt{x-1}}$ $\left[\text{form} \int \frac{x\,dx}{\text{Quadratic}\sqrt{\text{Linear}}}\right]$

Put $z = \sqrt{x-1} \Rightarrow z^2 = x-1 \Rightarrow 2z\,dz = dx$ Then

$$I = \int \frac{(z^2+1)(2z)dz}{\left\{(z^2+1)^2 - 2(z^2+1)+2\right\}(z)} = 2\int \frac{(1+z^2)dz}{z^4+2z^2+1-2z^2-2+2}$$

$$I = 2\int \frac{1+z^2}{1+z^4}dz = 2\int \frac{z^2\left(1/z^2\ +1\right)}{z^2\left(1/z^2\ +z^2\right)}dz = 2\int \frac{1+1/z^2}{1/z^2+z^2} \qquad (1)$$

Put $z-\frac{1}{z}=t \Rightarrow \left(1+\frac{1}{z^2}\right)dz = dt$ into (1), we get

$$I = 2\int \frac{dt}{t^2+2} = 2\int \frac{dt}{t^2+\left(\sqrt{2}\right)^2} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) = \sqrt{2}\tan^{-1}\left(\frac{z-1/z}{\sqrt{2}}\right) = \sqrt{2}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right)$$

$$I = \sqrt{2}\tan^{-1}\left(\frac{x-1-1}{\sqrt{2}\sqrt{x-1}}\right) = \sqrt{2}\tan^{-1}\left(\frac{x-2}{\sqrt{2x-2}}\right)+c.$$

(iii) Let $I = \int \frac{1dx}{(x+1)\sqrt{x^2-1}}$ $\left[\text{form} \int \frac{1dx}{\text{Linear}\sqrt{\text{Quadratic}}}\right]$

Put x + 1 = 1/z ➔ $dx = -dz/z^2$. Then

$$I = \int \frac{1}{(1/z)\sqrt{\left(\frac{1}{z}-1\right)^2-1}}\left(-\frac{1}{z^2}\right)dz = \int \frac{\left(-1/z^2\right)dz}{(1/z)\sqrt{\frac{1-2z+z^2-z^2}{z^2}}}$$

$$I = \int \frac{\left(-1/z^2\right)dz}{\left(1/z^2\right)\sqrt{1-2z}} = -\int \frac{dz}{(1-2z)^{1/2}} = -\int (1-2z)^{-1/2}dz$$

$$I = \frac{1}{2}\frac{(1-2z)^{1/2}}{1/2} = \sqrt{1-2z} = \sqrt{1-\frac{2}{x+1}} = \sqrt{\frac{x+1-2}{x+1}} = \sqrt{\frac{x-1}{x+1}}+c.$$

(iv) Let $I = \int \frac{1dx}{x^2\sqrt{x^2+1}}$ $\left[\text{form} \int \frac{dx}{\text{Quadratic}\sqrt{\text{Quadratic}}}\right]$

Put x = 1/y ➔ $dx = -1/y^2\ dy$. Then

$$I = \int \frac{\left(-1/y^2\right)dy}{(1/y)^2\sqrt{(1/y)^2+1}} = \int \frac{\left(-1/y^2\right)dy}{\left(1/y^2\right)\sqrt{\frac{1+y^2}{y^2}}} = -\int \frac{ydy}{\sqrt{1+y^2}} = -\frac{1}{2}\int \frac{2ydy}{\sqrt{1+y^2}}$$

$$I = -\frac{1}{2}\left[\frac{\left(1+y^2\right)^{-1/2+1}}{-1/2+1}\right] \qquad \left[\text{using} \int \frac{f'(x)dx}{\{f(x)\}^n} = \frac{\{f(x)\}^{-n+1}}{-n+1}\right]$$

$$I = -\sqrt{1+y^2} = -\sqrt{1+\left(\frac{1}{x}\right)^2} = -\sqrt{\frac{x^2+1}{x^2}} = -\frac{\sqrt{x^2+1}}{x}+c.$$

Integration of Rational Trigonometric Functions

In this section we shall learn integration of function of the form

$$F(x) = f(x)/g(x)$$

Here f(x) and g(x) may be any trigonometric function. In such cases we make the substitution: $z = \tan\frac{x}{2} \Rightarrow dz = \text{sex}^2\frac{x}{2}.\frac{1}{2}dx$ ➔ $dx = 2dz/\sec^2\frac{x}{2} = 2dz/\left(\tan^2\frac{x}{2}+1\right) = \frac{2dz}{(z^2+1)}$

Also $\tan\frac{x}{2} = z = \frac{z}{1} = \frac{P}{B} \Rightarrow H = \sqrt{P^2+B^2} = \sqrt{z^2+1}$ **NOTE: B = Base, P = Perp, H = Hyp**

$\Rightarrow \sin\frac{x}{2} = \frac{P}{H} = \frac{z}{\sqrt{z^2+1}}$ and $\cos\frac{x}{2} = \frac{B}{H} = \frac{1}{\sqrt{z^2+1}}$

Thus $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{z}{\sqrt{z^2+1}}\frac{1}{\sqrt{z^2+1}} = \frac{2z}{z^2+1}$

And $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{z^2+1} - \frac{z^2}{z^2+1} = \frac{1-z^2}{z^2+1}$

$\Rightarrow \tan x = \sin x / \cos x = 2z/(1-z^2)$

REMARK: In the following few examples that are related with the above topic we shall make direct use of these formulae.

Example 01: Evaluate the following integrals

(i) $\int \frac{1}{\sin x + \cos x}dx$ **(ii)** $\int \frac{\sin x}{1+\cos x}dx$ **(iii)** $\int \frac{\tan x}{1+\sin x}dx$

Solution: Making the substitutions as shown above, we have

$$\int \frac{1}{\sin x+\cos x}dx = \int \frac{1}{\left[2z/(z^2+1)+(z^2-1)/(z^2+1)\right]}\frac{2dz}{(z^2+1)}$$

$$= \int \frac{z^2+1}{\left[2z+(z^2-1)\right]}\frac{2dz}{(z^2+1)} = 2\int \frac{1}{z^2+2z-1}dz = 2\int \frac{1}{(z^2+2z+1)-2}dz$$

$$= 2\int \frac{1}{(z+1)^2-(\sqrt{2})^2}dz \qquad (1)$$

Substituting $z+1=u \Rightarrow dz = du$. Also let $\sqrt{2} = a \Rightarrow (\sqrt{2})^2 = a^2$. Thus (1) becomes:

$$\int \frac{1}{\sin x+\cos x}dx = 2\int \frac{1}{u^2-a^2}du = 2.\frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)+C = \frac{1}{\sqrt{2}}\ln\left(\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right)+C$$

Substituting $z = \tan(x/2)$, we get: $\int \frac{1}{\sin x+\cos x}dx = \frac{1}{\sqrt{2}}\ln\left(\frac{\tan(x/2)+1-\sqrt{2}}{\tan(x/2)+1+\sqrt{2}}\right)+C$

(ii) $$\int \frac{\sin x}{1+\cos x}dx = \int\left[\frac{2z}{z^2+1}\right] \div \left[1+\frac{1-z^2}{1+z^2}\right]\frac{2dz}{(z^2+1)} = \int\left[\frac{2z}{z^2+1}\right] \div \left[\frac{1+z^2+1-z^2}{1+z^2}\right]\frac{2dz}{(z^2+1)}$$

$$= \int\left[\frac{2z}{z^2+1}\right]\times\left[\frac{1+z^2}{2}\right]\frac{2dz}{(z^2+1)} = \int\left[\frac{2z}{z^2+1}\right] = \ln(z^2+1) = \ln\left(\tan^2\frac{x}{2}+1\right)+C$$

(iii) $$\int \frac{\tan x}{1+\sin x}dx = \int\left[\frac{2z}{1-z^2}\right] \div \left[1+\frac{2z}{1+z^2}\right]\frac{2dz}{z^2+1} = \int\left[\frac{2z}{1-z^2}\right] \div \left[\frac{1+z^2+2z}{1+z^2}\right]\frac{2dz}{z^2+1}$$

$$= \int\left[\frac{2z}{1-z^2}\right]\times\left[\frac{1+z^2}{(1+z)^2}\right]\frac{2dz}{z^2+1} = 4\int \frac{z}{(1-z)^2(z^2+1)}dz \qquad (1)$$

Now consider: $$\frac{z}{(1-z)^2(z^2+1)} = \frac{z}{(1-z)(1+z)(z^2+1)} = \frac{A}{1-z}+\frac{B}{1+z}+\frac{Cz+D}{z^2+1} \qquad (2)$$

➔ $z = A(1+z)(z^2+1)+B(1-z)(z^2+1)+(Cz+D)(1-z^2)$ (3)

Putting $1 - z = 0$ ➔ $z = 1$ in (3), we get:

$1 = A(2)(2)$ ➔ $A = 1/4$

Putting $1 + z = 0$ ➔ $z = -1$ in (3), we get

$-1 = B(-2)(2)$ ➔ $B = 1/4$

To find C and D we rewrite equation (3) as:

$z = A(z^3 + z^2 + z + 1) + B(-z^3 + z^2 - z + 1) + C(z - z^3) + D(1 - z^2)$

Equating the coefficients of:

z^3: $0 = A - B - C$ ➔ $C = A - B = 1/4 - 1/4 = 0$

z^2: $0 = A + B - D$ ➔ $D = A + B = 1/4 + 1/4 = 1/2$

Thus equation (2) becomes: $\dfrac{z}{(1-z)^2(z^2+1)} = \dfrac{1/4}{1-z}+\dfrac{1/4}{1+z}+\dfrac{1/2}{z^2+1}$

Integrating, we get: $\displaystyle\int\frac{z}{(1-z)^2(z^2+1)}dz = \frac{1}{4}\int\frac{1}{1-z}dz+\frac{1}{4}\int\frac{1}{1+z}dz+\frac{1}{2}\int\frac{1}{z^2+1}dz$

$$= -\frac{1}{4}\ln(1-z)+\frac{1}{4}\ln(1+z)+\frac{1}{2}\tan^{-1}z+C$$

Substituting $z = \tan(x/2)$, we get:

$$\int\frac{\tan x}{1+\sin x}dx = -\frac{1}{4}\ln\left(1-\tan\frac{x}{2}\right)+\frac{1}{4}\ln\left(1+\tan\frac{x}{2}\right)+\frac{1}{2}\tan^{-1}\left(\tan\frac{x}{2}\right)+C$$

Or $$\int\frac{\tan x}{1+\sin x}dx = -\frac{1}{4}\ln\left(1-\tan\frac{x}{2}\right)+\frac{1}{4}\ln\left(1+\tan\frac{x}{2}\right)+\frac{1}{2}\cdot\frac{x}{2}+C$$

$$\int\frac{\tan x}{1+\sin x}dx = \frac{1}{4}\left[-\ln\left(1-\tan\frac{x}{2}\right)+\ln\left(1+\tan\frac{x}{2}\right)+x+C\right]$$

8.3 APPLICATIONS OF INDEFINITE INTEGRATION

This section is devoted to present a variety of applications of anti-differentiation or indefinite-integration. In each example value of the constant of integration `c` is determined by given conditions imposed on the problem under discussion. When this is done, the resulting solution is called particular solution.

Example 01: A company finds that the marginal cost when x units of merchandise is produced is 50 – 0.08x dollars. If the fixed cost (overhead) is $700, determine

(a) the cost of producing x units

(b) the cost of producing 10 units

Solution: We know that marginal cost is the derivative of cost function. Using C(x) for the cost function, we have: $C'(x) = 50 - 0.08x$

Integrating both sides with respect to x, we get

$$C(x)=\int(50-0.08x)dx = 50x-\frac{0.08}{2}x^2+c = 50x-0.04x^2+c \quad (1)$$

Since fixed cost (overhead) is $700 which is the cost when no item is yet produced. In other words, C(0) = 700. This information is now used to find the constant c. Putting x = 0 and C = 700 in (1), we get:

$C(0)=50(0)-.04(0)^2+c \Rightarrow 700=0-0+c \Rightarrow c=700$. Thus,

a. $C(x) = 50x - 0.04x^2 + 700$ [using (1)]

b. $C(10) = 50(10) - 0.04(10)^2 + 700 = \$1196$.

Example 02: A toy rocket is shot vertically upward from the ground with an initial velocity of 300 feet per second. The acceleration due to gravity is -32 feet per sec^2.

(a) Find a formula for the rocket's velocity t seconds after the launch.

(b) Find a formula for the rocket's distance above the ground at any time t.

Solution: (a) We know that the acceleration is the derivative of velocity, so, a = dv/dt.
The acceleration is given as -32, so we have: dv/dt = -32
Integrating both sides with respect to t, we get

$$v = \int(-32)\,dt \Rightarrow v = -32t + C \qquad (1)$$

To determine C, use the fact that the initial velocity is 300 feet per second. This means v = 300 when t = 0. Substituting these values into (1) yields

$$300 = -32(0) + C \rightarrow C = 300$$

Thus, (1) becomes: v = -32 t + 300 [using (1)]

(b) We also know that the velocity is derivative of distance. That is;

$$v = ds/dt = -32t + 300$$

Integrating both sides with respect to t, we get

$$s = \int(-32t + 300)\,dt = -\frac{32}{2}t^2 + 300t + C \Rightarrow s = -16t^2 + 300t + C \qquad (2)$$

To find C, note that at the beginning (when t = 0) the rocket's distance s above the ground is zero, because it is shot upward from the ground. Substituting 0 for t and 0 for s into (2), we get: $0 = -16(0) + 300(0) + C \rightarrow C = 0$

Thus, (2) becomes: $s = -16t^2 + 300t$

Example 03: To test learning, a psychologist asks people to memorize a long sequence of digits. Assume that the rate at which digits are being memorized is dn/dt = 5.4 e$^{-0.3t}$, where `n` is the number of digits memorized and t is the time in minutes.

(a) Find `n` as a function of t, which gives the number of digits memorized after t minutes.

(b) How many digits will be memorized after 5 minutes?

Solution: (a) Here $dn/dt = 5.4\,e^{-0.3t}$
Integrating both sides with respect to t, we get

$$n = 5.4\int e^{-0.3t}dt = \frac{5.4}{-0.3}e^{-0.3t} + C \Rightarrow n = -18e^{-0.3t} + C \qquad (1)$$

To find C, we use the fact that at the beginning (when t = 0), the number of digits memorized is zero (that is, n = 0). Thus, putting n = 0 when t = 0 into (1), we get

$$0 = -18\,e^{0.3(0)} + C \rightarrow C = 18$$

Thus, the number of digits memorized (n) as a function of time (t) is

$$n = 18 - 18\,e^{-0.3t} = 18[1 - e^{-0.3t}]$$

(b) After 5 minutes (t = 5), the number of digits memorized is

$$n = 18 - 18e^{-0.3(5)} = n = 18 - 18e^{-1.5} = n = 18 - 18(.2231) \Rightarrow y \approx 14.$$

Thus approximately, 14 digits will be memorized after 5 minutes.

Example 04: Determine the profit function P(x) that corresponds to the given marginal profit: P`(x) = 70 – e$^{-0.01x}$, P(0) = -30 dollars

Solution: Here P`(x) = 70 – e$^{0.01x}$. Integrating both sides, we get

$$P(x) = \int(70 - e^{-0.01x})\,dx = 70x - \frac{1}{-0.01}e^{-0.01x} + C \Rightarrow P(x) = 70x + 100e^{-0.01x} + C \quad (1)$$

Put x = 0 and P = -30 into (1), we get

$$-30 = 70(0) + 100(e^0) + C \rightarrow C = -130$$

Thus, $P(x) = 70x + 100\,e^{-0.01x} - 130$

Example 05: A ball is shot vertically upward from the edge of a building with initial velocity 352 feet per second. The building is 768 feet tall. Acceleration due to gravity is -32feet per second per second.

(a) Determine the equations that describe the velocity of the ball and its distance from the ground.

(b) How far above the ground is the ball after 6 seconds and how fast is it going then?

Solution: (a) We know that the acceleration is the derivative of velocity. Therefore,

$$a = dv/dt = -32$$

Integrating both sides, we get: $v = -32\int 1\,dt = -32t + C$ (1)

To find C, use the fact that the initial velocity is 352 ft/sec. This means v = 352 when t = 0.

Thus, (1) becomes, $352 = -32(0) + C \Rightarrow C = 352$

Thus, we have $v = -32t + 352$ (2)

We know that velocity is the derivative of distance, that is;

$v = ds/dt = -32t + 352$ [from (2)]

Integrating both sides, we get

$$s = \int(-32t + 352)dt = -32\frac{t^2}{2} + 352t + C \Rightarrow s = -16t^2 + 352t + C \quad (3)$$

To find C, note that at the beginning (when t = 0) the ball's distance s from the ground is 768 ft, because it is shot upward from the building that is 768 feet above the ground.

Substituting 0 for t and 768 for s into (3), we get: 768 = C

Thus, $s = -16t^2 + 352t + 768$ (4)

Hence the equation that describes the velocity of the ball is $v = -32t + 352$ and its distance from the ground is: $s = -16t^2 + 352t + 768$

(b) At t = 6 sec, we have from (4), $s = -16(36) + 352(6) + 768 = 2304$ ft

Also from (2), we have $v = -32(6) + 352 = 544$ ft/sec.

Example 06: A tourist accidentally drops his camera from the top of a cliff that is 576 feet above the water below. Assume the acceleration due to gravity to be -32feet per second per second.

(a) Determine the velocity v(t) of the camera at any time t during its fall.

(b) Determine s(t), the height of the camera above the water at any time t during its fall.

(c) How fast is the camera falling 4 seconds after it is dropped?

(d) How long will it take the camera to hit the water?

Solution: (a) We know that: $a = dv/dt = -32$

Integrating both sides, we get: $v = \int(-32t)dt = -32t + C$ (1)

Now, when t = 0, v = 0. Thus, $0 = -32(0) + C \Rightarrow C = 0$

Thus (1) becomes: $v = -32t$ (2)

(b) The velocity is the derivative of distance, that is;

$v = ds/dt = -32t$ [from (2)]

Integrating both sides, we get: $s = \int(-32t)dt = -32\frac{t^2}{2} + C \Rightarrow s = -16t^2 + C$ (3)

Since it is given that s = 576 ft when t = 0. Thus (3) becomes,

$576 = -16(0) + C \Rightarrow C = 576$

$\Rightarrow s = -16t^2 + 576$ (4)

(c) At t = 4 sec, (2) becomes, $v = -32(4) = -128$

Thus, the camera is being fallen 128 feet/sec fast, 4 seconds after it is dropped.

(d) When the camera hits the water, its distance will vanish, so from (4) we have

$$0 = -16t^2 + 576 \Rightarrow t = 6$$

Thus, the camera will take 6 seconds to hit the water.

Example 07: On the moon the magnitude of the acceleration due to gravity is less than on the earth; it is approximately -5.3 feet per second per second. Consider a ball thrown upward from the surface of the moon with a velocity of 120 feet per second.

(a) Obtain a function that gives the velocity of the ball at any time t.

(b) Determine a function that shows the distance of the ball from the moon's surface at any time t.

Solution: (a) Given that acceleration $a = dv/dt = -5.3$

Integrating both sides with respect to t, we get

$$v = \int(-5.3)dt = -5.3\int 1\, dt \Rightarrow v = -5.3t + C \qquad (1)$$

Since the initial velocity of ball is 120 feet/sec, that is; when $t = 0$, $v = 120$. Thus (1) becomes:

$$120 = -5.3(0) + C \Rightarrow C = 120$$

Substituting C = 120 into (1), we get: $v = -5.3t + 120$

This is the required function that gives the velocity of the ball at any time t.

(b) We know that velocity is the derivative of distance, that is;

$$v = ds/dt = -5.3t + 120$$

Integrating both sides with respect to t, we get

$$s = \int(-5.3t + 120)dt = -2.65t^2 + 120t + C \qquad (2)$$

Initially the ball covers no distance, that is; when $t = 0$, $s = 0$. Substituting these values into (2), we get:

$$0 = -2.65(0) + C \Rightarrow C = 0$$

Now equation (2) becomes: $s = -2.65t^2 + 120t$

This is the required function that shows the distance of the ball from the moon's surface at any time t.

Example 08: The height h (in feet) of a tree is a function of time t(in years). Suppose you begin (t = 0) by planting a 5-foot tree in your yard and the tree grows to maturity according to the formula $\frac{dh}{dt} = 0.3 + \frac{1}{\sqrt{t}}$, $t > 0$

(a) Determine a formula for the height of the tree at any time t.

(b) Find the height of the tree after 1year, 4years, 9years, and 16years.

Solution: (a) Here $dh/dt = 0.3 + 1/\sqrt{t}$

Integrating both sides with respect to t, we get

$$h = \int\left(0.3 + \frac{1}{\sqrt{t}}\right)dt = 0.3t + \frac{t^{1/2}}{1/2} + C \Rightarrow h = 0.3t + 2\sqrt{t} + C \qquad (1)$$

At $t = 0$ and $h = 5$. Substituting these values into (1), we get

$$5 = 0.3(0) + 2\sqrt{0} + C \Rightarrow C = 5$$

Thus (1) becomes: $h = 0.3t + 2\sqrt{t} + 5 \qquad (2)$

which is the required formula for the height of the tree at any time t.

(b) When $t = 1$, $h = 0.3(1) + 2\sqrt{1} + 5 = 7.3$ ft

When $t = 4$, $h = 0.3(4) + 2\sqrt{4} + 5 = 10.2$ ft

When $t = 9$, $h = 0.3(9) + 2\sqrt{9} + 5 = 13.7$ ft, and

When $t = 16$, $h = 0.3(16) + 2\sqrt{16} + 5$ ft

Example 09: From the collected data, the health office estimates that a flu virus is spreading through the country at the rate of $5t^{2/3} + 22$ people per day.

(a) If n is the number of people who have the flu at any time t, where t is the time in days, complete the equation dn/dt= …………

(b) If 50 people had the flu at the beginning of the outbreak, determine an equation that expresses n as a function of t.

(c) How many people have the flu after 8 days?

Solution: (a) Here $dn/dt = 5t^{2/3} + 22$

Integrating, $n = \int\left(5t^{2/3} + 22\right)dt = 5\dfrac{t^{5/3}}{5/3} + 22t + C \Rightarrow n = 3t^{5/3} + C$ (1)

(b) Now n = 50 and t = 0, therefore, from (1) $50 = 3(0) + C$ ➔ $C = 50$

Thus (1) becomes $n = 3t^{2/3} + 50$ (2)

(d) When t = 8, from (2) $n = 3(8)^{2/3} + 50 = 146$. Thus, after 8 days 146 people have got the flu.

WORKSHEET 08

Evaluate the following:

1. $\int \dfrac{ax^2 + bx + c}{x^2}dx$
2. $\int \sqrt{x}\left(ax^2 + bx + c\right)dx$
3. $\int\left(x + \dfrac{1}{x}\right)^3 dx$
4. $\int\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^3 dx$
5. $\int\left(a^2 - x^2\right)^3 dx$
6. $\int\left(x + \dfrac{1}{x}\right)\left(x^2 + \dfrac{1}{x^2}\right)dx$
7. $\int \dfrac{x^3 + 3x^2 + 4}{\sqrt{x}}dx$
8. $\int \dfrac{dx}{\sqrt{x+1} - \sqrt{x-2}}$
9. $\int \dfrac{(2x+1)^3}{x+1}dx$
10. $\int \sin 2x\sqrt{1 - \cos 2x}\,dx$
11. $\int\left(e^{a\ln x} + e^{x\ln a}\right)dx$
12. $\int(1+x)\sqrt{1-x}\,dx$
13. $\int \dfrac{dx}{\sqrt{x^2 - 4}}$
14. $\int \dfrac{dx}{\sqrt{4 - x^2}}$
15. $\int \sqrt{\dfrac{1-x}{1+x}}dx$
16. $\int \dfrac{\sin(\ln x)}{x}dx$
17. $\int \dfrac{2x+7}{\sqrt{x^2 + 7x + 6}}dx$
18. $\int \dfrac{x+6}{\sqrt{x^2 + 12x + 7}}dx$
19. $\int \sin^5\theta \cos^3\theta\, d\theta$
20. $\int \sec^5\theta \tan^3\theta\, d\theta$
21. $\int \csc^5\theta \cot^3\theta\, d\theta$
22. $\int \sqrt{a^2 + x^2}dx$
23. $\int \sqrt{x^2 - a^2}dx$
24. $\int e^{\cos x}\sin x\, dx$
25. $\int \dfrac{\sin 2x}{\sqrt{1 + \cos^2 x}}dx$
26. $\int \dfrac{1}{\sqrt{x}}\operatorname{cosec}\sqrt{x}\cot\sqrt{x}\, dx$
27. $\int\left(3^{\sin x} + [\sin x]^3\right)\cos x\, dx$
28. $\int \dfrac{\sin x}{2\cos x + 7\sqrt{\cos x}}dx$
29. $\int x\operatorname{cosec}^2 x\, dx$
30. $\int x^n \ln x\, dx$
31. $\int \operatorname{cosec}^3 x\, dx$
32. $\int x\sin^{-1} x\, dx$
33. $\int e^{ax}\cos(bx + c)dx$
34. $\int \dfrac{\sin x}{4 + 5\cos x}dx$
35. $\int \dfrac{e^x - e^{-x}}{e^x + e^{-x}}dx$
36. $\int \dfrac{dx}{\sqrt{x}\left(1 + \sqrt{x}\right)}$
37. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}}$
38. $\int \dfrac{dx}{(2x+3)^2 + 1}$
40. $\int \dfrac{x+5}{\sqrt{1 - x^2}}dx$

41. $\int \frac{e^{\sqrt{x}}\cos\left(e^{\sqrt{x}}\right)}{\sqrt{x}}dx$ 42. $\int \frac{\ln\left(1+x^2\right)}{x^2}dx$ 43. $\int x\cot^{-1}x\,dx$

44. $\int x^3e^{x^2}dx$ 45. $\int \sin(\ln x)dx$ 46. $\int e^x\sec x(1+\tan x)dx$

47. $\int \frac{\sin(\ln x)}{x^3}dx$ 48. $\int \frac{dx}{2x^2-x-1}$ 49. $\int \frac{2x-3}{\left(x^2-1\right)(2x+3)}dx$

50. $\int \frac{x}{x^2-12x+35}dx$ 51. $\int \frac{dx}{1+3e^x+2e^{2x}}$ 52. $\int \frac{3x+1}{(x-1)^2(x+3)}dx$

53. $\int \frac{x\,dx}{(x-1)\left(x^2+4\right)}$ 54. $\int \frac{xdx}{x^4-x^2-1}$ 55. $\int \frac{1-x^2}{\left(1+x^2\right)^2}dx$

56. $\int \sqrt{5-4x-x^2}dx$ 57. $\int \frac{dx}{\sqrt{2x^2+3x+4}}$ 58. $\int \frac{\cos x dx}{\sqrt{4\sin^2 x+4\sin x+5}}$

59. $\int \frac{dx}{\left(x^2+1\right)\sqrt{x}}$ 61. $\int \frac{dx}{\left(2ax+x^2\right)^{3/2}}$ 62. $\int \frac{(x+1)(x+2)}{\sqrt{x^2+x-2}}dx$

63. $\int \frac{\sin x}{(1+\cos x)(2+\cos x)}dx$ 64. $\int \frac{x^2}{(1+x)\left(x^2-4x+1\right)}dx$

65. $\int \frac{\cos x}{(\sin x+\cos x)}dx$ 66. $\int \frac{\sin x}{(\sin x-\cos x)}dx$ 67. $\int \frac{\tan x}{1+\cos x}dx$

68. A tourist accidentally drops his camera from the top of a cliff that is 576 feet above the water below. Assume the acceleration due to gravity to be -32 feet per second per second.
(a) Determine the velocity v(t) of the camera at any time t during its fall.
(b) Determine s(t) the height of the camera above the water at any time t during its fall.
(c) How fast is the camera falling 4 seconds after it is dropped?
(d) How long will it take the camera to hit the water?
(Hint: What is the value of s when the camera hits the water?)

69. To test learning, a psychologist asks people to memorize a long sequence of digits. Assume that the rate at which digits are being memorized is $dy/dt = 5.4e^{-0.3t}$ words per minute, where y is the number of digits memorized and t is the time in minutes.
(a) Find y as a function of t, which gives the number of digits memorized after t minutes.
(b) How many digits will be memorized after 5 minutes?

70. The rate of change of the volume of a spherical balloon with respect to its radius is $dV/dr = 4\pi r^2$. Use this fact and the fact that V = 0 when r = 0, determine the volume of the balloon when its radius is 6 centimeters.

71. If the marginal cost when x units is produced is C(x0 = 100 – 0.5x dollars and the overhead cost is $40, what is the cost of producing 10 units?

72. Let R(x) be the revenue a company receives from the sale of x units of its product. If its marginal revenue is **R`(x) = 100 – 0.2x** dollars per item, find the:
(a) Revenue function R(x) and the revenue from the sale of 20 units. Assume there is no revenue when zero units are sold.

73. A woman gets into her car and then drives it with a constant acceleration of 22 feet per second per second.
(a) Determine the velocity function.

(b) Determine the distance function.

(c) How far does the car go in 6 seconds?

74. A colony of 200 bacteria is introduced to a growth-inhibiting environment and grows at the rate of dn/dt = 30 + 2t, where n is the number of bacteria present at any time t (t is measured in hours).

(a) Determine a function that gives the number of bacteria present at any time t.

(b) How many bacteria are present after 3 hours?

75. The weight of a mold is growing exponentially at the rate of dw/dt = $e^{0.2t}$ milligrams per hour. How much will the mold weigh in 10 hours if it weighs 70milligrams now?

73. The rate of change of the temperature T inside a furnace after x minutes is dT/dx = 2x + 15 (1 < x < 20) degree/min. Assume the temperature inside the furnace is 200°F initially.

(a) Find the formula for the temperature at any time x.

(b) What is the temperature inside the furnace after 14 minutes?

76. After t hours of production, a coal mine is producing coal at the rate of dP/dt = 30 + 2t – 0.03t tons per hour. Find a formula for the total output of the coal mine after t hours of production. (Note: Coal production P = 0 at t = 0).

77. The rate at which atmospheric pressure P changes as the height x above sea level changes is dP/dx = -3.087 $e^{-0.21x}$ measured in pounds per square inch and x is in miles. Determine P as a function of x when at sea level, P is 14.7 pounds per square inch.

78. The rate of the change of the area of a circular region with respect to its radius is dA/dr = 2πr. Use this fact, and the fact that A = 0 when r = 0, to determine the area of a circular region when the radius is 4 centimeters.

79. The rate of change of the volume of a spherical balloon with respect to its radius is dV/dr = 4 πr^2. Use this fact, and the fact that V = 0 when r = 0, to determine volume of the balloon when its radius is 6 centimeters.

80. Flu epidemic is spreading at the rate dn/dt = 180t – $6t^2$, where n is the number of people who are sick with flu on any particular day t after the outbreak started.

(a) Determine an equation for n as a function of t. Assume no one has the flu at the beginning (when t = 0).

(b) How many people have the flu the tenth day after the outbreak begins?

81. Plaque builds up on the inside walls of an artery reduces the diameter of the artery (and thus reduces the blood flow). Suppose that an artery has a diameter of .4 centimeter and the length of diameter (D) is decreasing at the rate of:

dD/dt = -0.03$e^{-0.001t}$ cm/year.

(a) Determine D as a function of t.

(b) What will the diameter of the artery be after 10 years?

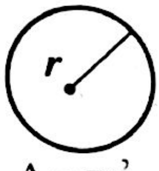
82. A manufacturer estimates that marginal cost of a certain production process is given by $C'(x) = e^{0.01x} + 3\sqrt{x}$, where x units are produced. What does it cost to produce 10 units if the cost of producing 4 units is $ 2000?

CHAPTER NINE

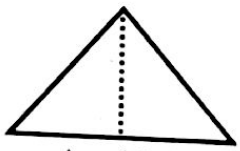
DEFNITE INTEGRATION

9.1 INTRODUCTION

The study of geometry includes formulas for determining the area bounded by geometric figures such as circles, triangles and rectangles shown as under.



$A = \pi r^2$

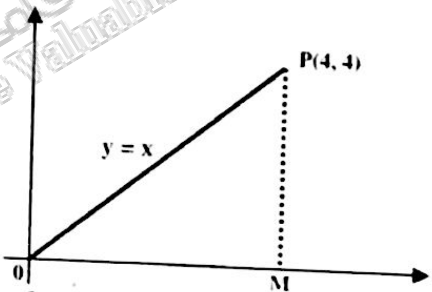


$A = bh / 2$

$A = l\,w$

There are occasions when we are asked to find the area under given curve. There are very few cases when it is possible to find this area by using simple techniques but in many cases it is not possible to find this area due to the nature of the curve. For example, consider an equation y = x. Its graph for values of x from 0 to 4 is shown as under:

x:	0	1	2	3	4
y:	0	1	2	3	4



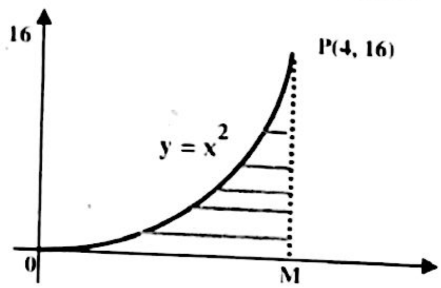
At point P(4, 4) draw a perpendicular PM on the x-axis. Thus OMP is a right triangle and its are is

$A = ½ \text{(Base) (Height)} = ½ (4) (4) = 8$

Thus area under the curve/line y = x between x = 0 and x = 4 is 8 unit^2.

Now consider an equation $y = x^2$. Its graph for values of x from 0 to 4 is shown as under:

x:	0	1	2	3	4
y:	0	1	4	9	16



At point P(4, 16) draw a perpendicular PM on the x-axis. Now OMP is not a right triangle hence area under the curve $y = x^2$ between x = 0 and x = 4 can not be determined using above technique.

Now a question arises that, is there any method to determine this area. The answer is yes.

We can determine this area with the help of **"Definite Integration"**.

Definition: If f(x) = F `(x), we define **definite integration** of f(x) between the limits x = a and x = b as: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

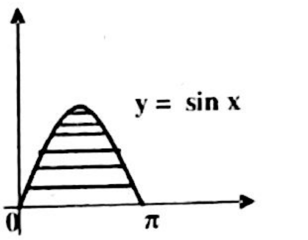
This is known as **"Fundamental Theorem of Calculus"**.

Using this theorem, we see that area under the curve $y = x^2$ between x = 0 and x = 4 is

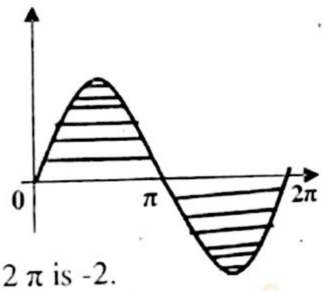
$$\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{1}{3}\left[4^3 - 0^3\right] = \frac{64}{3}\ \text{unit}^2$$

Let us consider the curve y = sin x. The area under this curve between x = 0 and x = π is given by:

$$\int_0^{\pi} \sin x \, dx = -\cos x\Big|_0^{\pi} = -(\cos\pi - \cos 0) = -(-1-1) = 2 \text{ unit}^2.$$



This area is shown in the adjacent figure.
Let us consider the same curve y = sin x but now we take the Values of x from x = 0 to x = 2π. The area is shown in the adjacent figure.



Thus $\int_0^{2\pi} \sin x \, dx = -\cos x\Big|_0^{2\pi} = -(\cos 2\pi - \cos 0) = -(1-1) = 0 \text{ unit}^2$

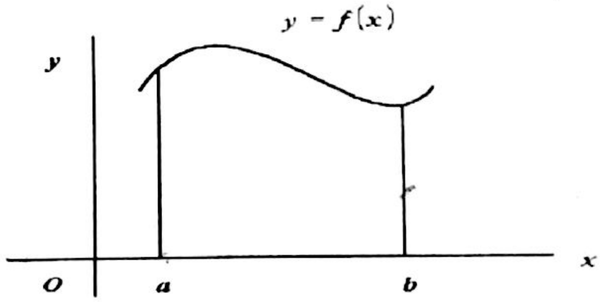
Here total area under the curve is based on two parts:
The area under the upper part of the curve y = sin x from x = 0 to x = π is 2. This is shown in the adjacent figure.
The area under the lower part of the curve y = sin x from x = π to x = 2 π is -2. (This is due to symmetry). Thus total area under the curve y = sin x from x = 0 to x = 2 π is (2 – 2) = 0. This gives a concept of negative area though an area cannot be negative. But in definite integration, we are evaluating an integration over the interval [a, b] whose value may be negative.
Thus one of the applications of definite integration is to find the area under the curve y = f(x) between the limits x = a and x = b.
In this chapter, we will develop the calculus necessary to determine the area under any curve y = f(x) or the area between two curves. We shall also study various applications of definite integration. But for time being, let us study the most important topic of integral calculus known as "Riemann Sums".

Definite Integration as Riemann Sum

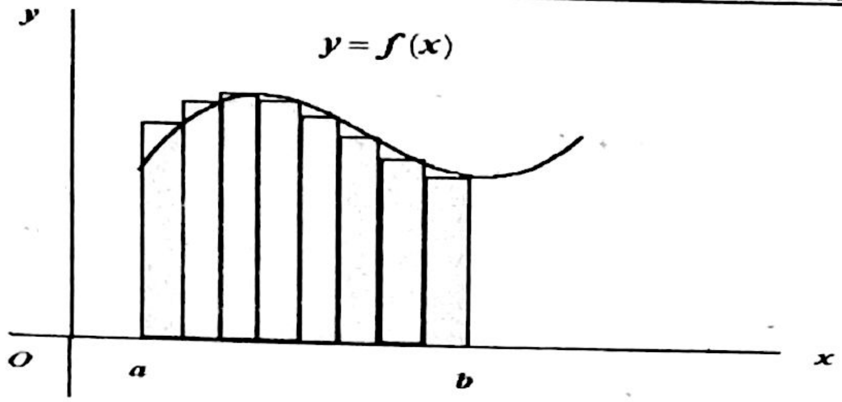
Consider an arbitrary function f(x) defined on a closed interval [a, b]. We shall assume the following:

i. f(x) is continuous on [a, b]
ii. $f(x) \geq 0$ for all x between a and b.

At this time we have no means of determining the exact value of the shaded area as shown in the figure below.



To determine this area we divide the interval [a, b] in n equal sub-interval and construct the rectangles, because it is an easy matter to determine the area of a rectangle. (The rectangles may be of variable widths but for sack of simplicity we consider each rectangle of same width). This is known as rectangular approximation of area under the given curve. The width of each rectangle therefore is $\Delta x = (b - a)/n$.
The height of each rectangle will be the distance between the x – axis and the graph, measured vertically at the right end of each subinterval. This is illustrated in the figure given below.



The approximation to the area under the curve y = f(x) improves as the number of rectangles increases further. We find the area of `n` rectangles to be

$$A = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Here $a = x_0 < x_1 < x_2 < \dots < x_n = b$. The exact area between the graph of y = f(x) and the x – axis (on the interval from a to b) is the limit of this sum as the number of rectangles `n` approaches infinity. Thus, assuming the limit exists:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Saying that $n \to \infty$ is the same as saying that $\Delta x \to 0$ since the width of each rectangle gets smaller and smaller as the number of rectangles increases. Thus, we can also write

$$A = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

This limit is given a special name and notation. It is known as the **definite integral** of function f(x) from x = a to x = b and is usually denoted by:

$$A = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

REMARK: The numbers a and b are called **limits of integration,** b is the **upper limit** and a is the **lower limit**.

Example 03: Evaluate $\int_1^2 x^2 dx$ using the Fundamental Theorem of Calculus.

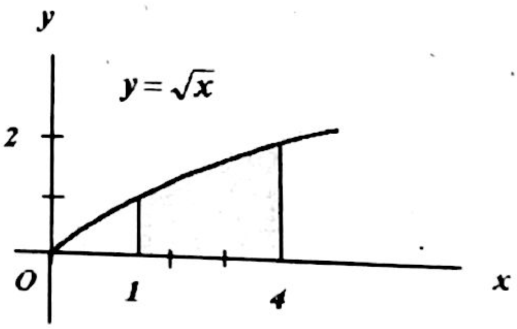
Solution: $\int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}\left[x^3\right]_1^2 = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}$

Example 04: Determine the (exact) area under the curve $y = \sqrt{x}$ from x = 1 to x = 4.

Solution: We have $f(x) = \sqrt{x}, a = 1$ and $b = 4$. Thus,

$$\text{Area} = \int_1^4 \sqrt{x}dx = \int_1^4 x^{1/2}dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4 = \left[\frac{2}{3}x^{3/2}\right]_1^4$$

$$= \frac{2}{3}\left(4^{3/2} - 1\right) = \frac{2}{3}(7) = \frac{14}{3}.$$



The area under $y = \sqrt{x}$ from x = 1 to x = 4

The area between the graph of $y = \sqrt{x}$ and the x – axis on the interval [1, 4] has been shown to be

14/3 square units. This area is shown shaded in the above figure.

Fundamental Properties of Definite Integrals

In this section, we shall discuss some important properties of definite integration which will help to solve very important problems of integration.

Property 1: $\int_a^b f(x)dx = \int_a^b f(u)du$ **[P-1]**

This property states that definite integral is independent of the variable used.

Proof: Let $\int f(x)dx = F(x) \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$. Similarly

$$\int f(u)du = F(u) \Rightarrow \int_a^b f(u)du = F(b) - F(a) \quad \Rightarrow \int_a^b f(x)dx = \int_a^b f(u)du$$

For example, $\int_0^{\pi/2} \cos x\, dx = \int_0^{\pi/2} \cos u\, du = 1$

Property 2: If f(x) is an integrable function, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **[P-2]**

This property states that interchange in the limits of integration changes the sign of the integral.

Proof: Let $\int f(x)dx = F(x) \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$ and

$$\int_b^a f(x)dx = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f(x)dx$$

For example, $\int_0^{\pi/2} \cos x\, dx = 1$ whereas $\int_{\pi/2}^{0} \cos x\, dx = -1$

Property 3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$ **[P-3]**

This property states that if c is any real number that lies in the interval [a, b] then integral from a to b is equal to sum of the integrals taken from a to c and then from c to b.

Proof: Let, $\int f(x)dx = F(x) \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$

$$\int_a^c f(x)dx = F(c) - F(a) \text{ and } \int_c^b f(x)dx = F(b) - F(c)$$

Now $\int_a^c f(x)dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f(x)dx$

For example, $\int_0^{\pi} \sin x\, dx = -[\cos x]_0^{\pi} = -(\cos\pi - \cos 0) = -(-1-1) = 2$

Now consider, $\int_0^{\pi} \sin x\, dx = \int_0^{\pi/2} \sin x\, dx + \int_{\pi/2}^{\pi} \sin x\, dx = -\left\{[\cos x]_0^{\pi/2} + [\cos x]_{\pi/2}^{\pi}\right\}$

$$= -[\cos(\pi/2) - \cos 0 + \cos\pi - \cos(\pi/2)] = -[0-1-1-0] = 2$$

We see that two results are same.

Generalization of Property 3

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)dx + \int_{a_n}^{b} f(x)dx$$

Here, $a < a_1 < a_2 < \ldots < a_n < b$.

This means that if the interval of integration [a, b] is divided into any finite number of subintervals, the integration taken over the interval [a, b] is equal to sum of the integrals taken over the subintervals.

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **[P-4]**

This property states that if the lower limit in any definite integral is zero, the value of the integral remains unchanged on replacing x by "upper limit minus x "in the integrand.

Proof: Let $u = a - x \Rightarrow du = -dx$. Also if $x = 0$ the $u = a$, and if $x = a$ then $u = 0$. Thus

$$\int_0^a f(a-x)dx = -\int_a^0 f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx$$ (Using properties 1 and 2)

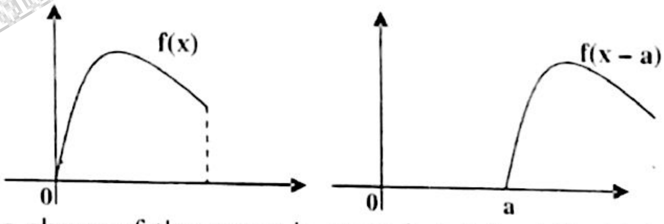
For example, consider, $\int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2} = \sin(\pi/2) - \sin 0 = 1 - 0 = 1$

Now consider, $\int_0^{\pi/2} \cos x\, dx = \int_0^{\pi/2} [\cos(\pi/2) - x]dx = \int_0^{\pi/2} \sin x\, dx = -[\cos x]_0^{\pi/2} = 1$

We observe that the result of both integrals is same.

REMARK: It may be noted that $\cos[(\pi/2) - x] = \sin x$.

Geometrically this property states that "area under the curve $y = f(x)$ from $x = 0$ to any real number $x = a$ is same as if the curve is shifted `a` units to right or left from the origin. This is depicted in the following figure.



We observe that the shape of the curve is same but it is shifted `a` units to the right of the origin. Thus area under both curves will be same.

Property 5: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$ **[P-5]**

Proof: By property 3, $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

But $\int_a^{2a} f(x)dx = \int_0^a f(2a-x)dx$. [Students may verify this by letting $u = 2a - x$].

Thus, $$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Property 6: i. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x)=f(x)$ **[P-6]**

ii. $\int_0^{2a} f(x)dx = 0$, if $f(2a-x)=-f(x)$

Proof : (i) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$ [By P−4]

$= \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$ $[\because f(2a-x)=f(x)]$

(ii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$ [By property 4]

$= \int_0^a f(x)dx - \int_0^a f(x)dx = 0$ $[\because f(2a-x)=-f(x)]$

For example, consider $\int_0^{2\pi} \cos x\,dx = [\sin x]_0^{2\pi} = \sin 2\pi - \sin 0 = 0-0=0$

And $\int_0^{\pi} \cos x\,dx + \int_0^{\pi} \cos(2\pi - x)dx = \int_0^{\pi} \cos x\,dx + \int_0^{\pi} \cos x\,dx = 2\int_0^{\pi} \cos x\,dx$

$= 2[\sin x]_0^{\pi} = 2(\sin\pi - \sin 0) = 0$. Thus LHS = RHS **NOTE: $\cos(2\pi - x) = \cos x$,**

(ii) We know that $\sin(2\pi - x) = -\sin x$, hence

$\int_0^{2\pi} \sin x\,dx = -[\cos x]_0^{2\pi} = -(\cos 2\pi - \cos 0) = -(1-1) = 0$.

Property 7: If $f(x) = f(a+x)$, then $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ **[P-7]**

Proof: $\int_0^{na} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx + \int_{2a}^{3a} f(x)dx + \ldots + \int_{(n-1)a}^{na} f(x)dx$ (1)

In the 2nd integral on the right hand side substitute, $x = a + u \Rightarrow dx = du$.
Now if $x = a$ then $u = 0$ and if $x = 2a$ then $u = a$. Thus,

$\int_a^{2a} f(x)dx = \int_0^a f(a+u)du = \int_0^a f(a+x)dx = \int_0^a f(x)dx$

Similarly, in the 3rd integral on the right side of (1) if we make the same substitution, we get:

$\int_{2a}^{3a} f(x)dx = \int_{0}^{a} f(x)dx$.

Proceeding in this manner we shall see that each integral on the right side of (1) is equal to $\int_0^a f(x)dx$.

Thus, $\int_0^{na} f(x)dx = n\int_0^{a} f(x)dx$ provided $f(x) = f(a+x)$.

For example,

$$n\int_0^{2\pi} \cos x\,dx = n[\sin x]_0^{2\pi} = n(\sin 2\pi - \sin 0) = n(0-0) = 0 \quad [\because \cos(2\pi - x) = \cos x]$$

$$\int_0^{2n\pi} \cos x\,dx = n\int_0^{2\pi} \cos x\,dx = n[\sin x]_0^{2\pi} = n(\sin 2\pi - \sin 0) = n(0-0) = 0$$

We observe that LHS = RHS.

Property 8:

[P-8]

- If f(x) is an even function of x, then: $\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx$
- If f(x) is an odd function of x, then: $\int_{-a}^{a} f(x)dx = 0$

Proof: Let f(x) be an even function, that is, f(-x) = f(x). Then

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^{a} f(x)dx = \int_{-a}^{0} f(-x)dx + \int_0^{a} f(x)dx, \text{ since } f(-x) = f(x).$$

Put –x = u in the 1st integral then dx = -du. If x = -a then u = a and if x = 0 then u = 0.
Thus,

$$\int_{-a}^{a} f(x)dx = -\int_{a}^{0} f(u)du + \int_0^{a} f(x)dx = \int_0^{a} f(u)du + \int_0^{a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(x)dx = 2\int_0^{a} f(x)dx$$

Now let f(x) be an odd function, that is, f(-x) = - f(x). Then

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^{a} -f(x)dx, \text{ since } f(-x) = -f(x).$$

Put –x = u in the 1st integral then dx = -du. If x = -a then u = a and if x = 0 then u = 0.

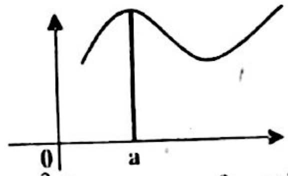
Thus, $\int_{-a}^{a} f(x)dx = -\int_{a}^{0} f(u)du - \int_0^{a} f(x)dx = \int_0^{a} f(u)du - \int_0^{a} f(x)dx$

$$= \int_0^{a} f(x)dx - \int_0^{a} f(x)dx = 0.$$ This proves the result.

REMARK: If f(x) is some function of x such that f(-x) = f(x) then f(x) is known as even function and if f(-x) = - f(x) then f(x) is known as odd function. If these conditions are not satisfied, then function f(x) is neither even nor odd.

For example, let $f(x) = x^2 + \cos x$ then:
$f(-x) = (-x)^2 + \cos(-x) = x^2 + \cos x = f(x)$. Thus f(x) is even function.
Now consider $f(x) = x^3 + \sin x$, then:
$f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x = -(x^3 + \sin x) = -f(x)$. Thus f(x) is odd function.
Now consider $f(x) = x^3 + 1$, then:
$f(-x) = (-x)^3 + 1 = -x^3 + 1 \neq f(x)$ hence, f(x) is not even function. Moreover,
$f(-x) = (-x)^3 + 1 = -x^3 + 1 = -(x^3 - 1) \neq -f(x)$ hence, f(x) is not odd function.
Thus $f(x) = x^3 + 1$ is neither even nor odd function.
Also, cos(-x) = cos x and sin(-x) = - sin x.

REMARK: $\int_a^a f(x)\,dx = 0$. Geometrically this means that area under a vertical line x = a is zero.



Example 05: We know that $f(x) = x^2$ is an even function and $f(x) = x^3$ is an odd function. Hence we may see that

$$\int_{-4}^{4} x^2dx = \left[\frac{x^3}{3}\right]_{-4}^{4} = \frac{1}{3}[64-(-64)] = \frac{128}{3}. \text{Also } 2\int_0^4 x^2dx = 2\left[\frac{x^3}{3}\right]_0^4 = \frac{2}{3}(64-0) = \frac{128}{3}$$

This shows that LHS = RHS.

$$\text{Also, } \int_{-4}^{4} x^3dx = \left[\frac{x^4}{4}\right]_{-4}^{4} = \frac{1}{3}(256-256) = 0$$

With the help of these properties we are now solving some important definite integrals.

Example 06: Evaluate the following:

(i) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ (ii) $\int_0^{\pi/2} \ln(\sin x)\,dx$ (iii) $\int_0^{\pi/2} \ln(\tan\theta+\cot\theta)\,d\theta$ (iv) $\int_0^{\pi/4} \ln(1+\tan x)\,dx$

Solution: (i) Using P-4;

$$\text{let, } I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

Adding the two results, we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_0^{\pi/2} 1dx = [x]_0^{\pi/2} = \left[\frac{\pi}{2}-0\right] = \frac{\pi}{2} \quad \Rightarrow I = \frac{\pi}{4}$$

$$\text{Thus, } I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \frac{\pi}{4}$$

(ii) Using property 4, let $I = \int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx = \int_0^{\pi/2} \ln(\cos x)\,dx$

NOTE: $\sin(\pi/2 - x) = \cos x$. Thus,

$I = \int_0^{\pi/2} \ln(\sin x)\,dx$ and $I = \int_0^{\pi/2} \ln(\cos x)\,dx$. Hence,

$$2I = \int_0^{\pi/2} \ln(\sin x)\,dx + \int_0^{\pi/2} \ln(\cos x)\,dx = \int_0^{\pi/2} [\ln(\sin x)+\ln(\cos x)]\,dx$$

$$= \int_0^{\pi/2} \ln(\sin x\cos x)\,dx = \int_0^{\pi/2} \ln\left(\frac{1}{2}2\sin x\cos x\right)dx = \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right)dx \quad \textbf{[2sinx cos x = sin 2x]}$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\,dx - \int_0^{\pi/2} \ln 2\,dx = \int_0^{\pi/2} \ln(\sin 2x)\,dx - \ln 2\int_0^{\pi/2} 1\,dx \quad \textbf{[ln(a/b) = lna - ln b]}$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\,dx - \ln 2[x]_0^{\pi/2} = \int_0^{\pi/2} \ln(\sin 2x)\,dx - \frac{\pi}{2}\ln 2$$

Thus, $I = \int_0^{\pi/2} \ln(\sin x)\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(\sin 2x)\,dx - \frac{\pi}{4}\ln 2$ (1)

Now putting $2x = u$ ➔ $2\,dx = du$ ➔ $dx = du/2$.
Also, if $x = 0$ then $u = 0$ and if $x = \pi/2$ ➔ $u = \pi$.

Thus, $\int_0^{\pi/2} \ln(\sin 2x)\,dx = \frac{1}{2}\int_0^{\pi} \ln(\sin u)\,du = \frac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx$ **[Using P-1]**

Now $\sin x = \sin(\pi - x)$. Thus using property 6, we get

$$\int_0^{\pi/2} \ln(\sin 2x)\,dx = \frac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx = \frac{1}{2}2\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\sin x)\,dx = I$$

Thus (1) becomes,

$$I = \int_0^{\pi/2} \ln(\sin x)\,dx = \frac{1}{2}I - \frac{\pi}{4}\ln 2 \Rightarrow I - \frac{1}{2}I = -\frac{\pi}{4}\ln 2 \Rightarrow \frac{I}{2} = -\frac{\pi}{4}\ln 2$$

$$➔\ I = \int_0^{\pi/2} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2$$

(iii) $I = \int_0^{\pi/2} \ln(\tan\theta + \cot\theta)\,d\theta = \int_0^{\pi/2} \ln\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)d\theta = \int_0^{\pi/2} \ln\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)d\theta$

$$= \int_0^{\pi/2} \ln\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta = \int_0^{\pi/2}\left[\ln(1) - \ln(\sin\theta\cos\theta)\right]d\theta = -\int_0^{\pi/2}\left[\ln\sin\theta + \ln\cos\theta\right]d\theta$$

$$= -\int_0^{\pi/2} \ln\sin\theta\,d\theta - \int_0^{\pi/2} \ln\cos\theta\,d\theta = -2\int_0^{\pi/2} \ln\sin\theta\,d\theta \qquad [\ln 1 = 0]$$

$$\left[\text{NOTE: } \int_0^{\pi/2} \ln\cos\theta\,d\theta = \int_0^{\pi/2} \ln\cos\left(\frac{\pi}{2} - \theta\right)d\theta = \int_0^{\pi/2} \ln\sin\theta\,d\theta\right]$$

Thus, $I = -2\left(-\frac{\pi}{2}\ln 2\right) = \pi\ln 2$ **[See part (ii)]**

(iv) Let $I = \int_0^{\pi/4} \ln(1 + \tan x)\,dx = \int_0^{\pi/4} \ln\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx$ [Using property 4]

$$= \int_0^{\pi/4} \ln\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx = \int_0^{\pi/4} \ln\left[1 + \frac{\tan(\pi/4) - \tan x}{1 + \tan(\pi/4)\tan x}\right]dx$$

$$= \int_0^{\pi/4} \ln\left[1 + \frac{1 - \tan x}{1 + \tan x}\right]dx = \int_0^{\pi/4} \ln\left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right]dx$$

$$= \int_0^{\pi/4} \ln\left[\frac{2}{1 + \tan x}\right]dx = \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan x)]\,dx$$

$$= \ln 2\int_0^{\pi/4} 1dx - \int_0^{\pi/4} \ln(1+\tan x)\,dx = \ln 2\left[x\right]_0^{\pi/4} - I = \frac{\pi}{4}\ln 2 - I$$

Thus $2I = \frac{\pi}{4}\ln 2 \Rightarrow I = \frac{\pi}{8}\ln 2$. Thus, $I = \int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

REMARK: The following formulae may be noted.

(a) $\sin\left(\frac{\pi}{2}-x\right) = \cos x$ **(b)** $\tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}$ **(c)** $\tan\left(\frac{\pi}{2}-x\right) = \cot x$

(d) $\sin(\pi - x) = \sin x$ **(e)** $\cos(\pi - x) = -\cos x$ **(f)** $\tan(\pi - x) = -\tan x$

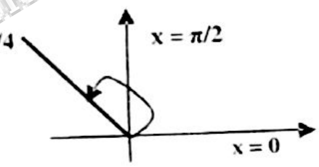
Example 07: Evaluate the following integrals

(i) $\int_0^4 f(x)\,dx$, where $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x-2 & \text{if } x > 2 \end{cases}$

Solution: $\int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^4 (x-2)\,dx$

$$\int_0^4 f(x)\,dx = \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 = \frac{8}{3} + \left[\frac{16-4}{2} - 2(4-2)\right] = \frac{8}{3} + 6 - 4 = \frac{8}{3} + 2 = \frac{14}{3}$$

(ii) $\int_0^{3\pi/4} |\cos x|\,dx$



Solution: $\int_0^{3\pi/4} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{3\pi/4} |\cos x|\,dx$

Now we know that in the 1st quadrant cos x is always positive and in the 2nd quadrant it is negative. Hence,

$$\int_0^{3\pi/4} |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/4} -\cos x\,dx = \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{3\pi/4}$$

$$= \left[\sin \pi/2 - \sin 0\right] - \left[\sin 3\pi/4 - \sin \pi/2\right] = 1 - 0 - 1/\sqrt{2} - 1 = -1/\sqrt{2}$$

(iii) $\int_0^{\pi} \cos^{2n+1} x\,dx$

Solution: $\int_0^{\pi} \cos^{2n+1} x\,dx = \int_0^{\pi} \cos^{2n} x \cos x\,dx = \int_0^{\pi} \left(\cos^2 x\right)^n \cos x\,dx = \int_0^{\pi} \left(1 - \sin^2 x\right)^n \cos x\,dx$

Putting $z = \sin x$ ➔ $dz = \cos x\,dx$.

If $x = 0$ ➔ $z = \sin 0 = 0$ and if $x = \pi$ ➔ $z = \sin \pi = 0$. Thus,

$$\int_0^{\pi} \cos^{2n+1} x\,dx = \int_0^0 \left(1 - z^2\right)^n dz = 0.$$

Notice that upper and lower limits of integration are same.

(iv) $\int_0^{\pi/2} \ln(\tan x)\,dx$

Solution: Let $I = \int_0^{\pi/2} \ln(\tan x)\,dx$. Using property 4, we have

$$I=\int_0^{\pi/2}\tan x\ln(\sin x)dx=\int_0^{\pi/2}\ln\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx=\int_0^{\pi/2}\ln(\cot x)dx$$

$$=\int_0^{\pi/2}\left[\ln(\tan x)\right]^{-1}dx=-\int_0^{\pi/2}\left[\ln(\tan x)\right]dx.$$

Thus, $I+I=2I=\int_0^{\pi/2}\ln(\tan x)dx-\int_0^{\pi/2}\ln(\tan x)dx=0$ ➔ $I=\int_0^{\pi/2}\ln(\tan x)dx=0$

(v) $\int_0^{\pi/2}\sin 2x\ln(\tan x)dx$

Solution: Let $I=\int_0^{\pi/2}\sin 2x\ln(\tan x)dx$. Using **[P-4]**

$$I=\int_0^{\pi/2}\sin 2\left(\frac{\pi}{2}-x\right)\ln\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx=\int_0^{\pi/2}\sin(\pi-2x)\ln(\cot x)dx$$

$$=\int_0^{\pi/2}\sin 2x\ln(\tan x)^{-1}dx=-\int_0^{\pi/2}\sin 2x\ln(\tan x)dx$$

Thus $I+I=\int_0^{\pi/2}\sin 2x\ln(\tan x)dx-\int_0^{\pi/2}\sin 2x\ln(\tan x)dx=0\Rightarrow 2I=0$

➔ $I=\int_0^{\pi/2}\sin 2x\ln(\tan x)dx=0$

(vi) $\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx$

Solution: Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx$. Using **P-4**

$$I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\cos(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)(-\tan x)}{(-\sec x)+(-\cos x)}dx$$

$$=-\pi\int_0^{\pi}\frac{\tan x}{-(\sec x+\cos x)}dx+\int_0^{\pi}\frac{x\tan x}{-(\sec x+\tan x)}dx=\pi\int_0^{\pi}\frac{\tan x}{(\sec x+\cos x)}dx-I$$

➔ $2I=\pi\int_0^{\pi}\frac{\tan x}{(\sec x+\cos x)}dx.$

Notice that: $\tan(\pi-x)=-\tan x$, $\sec(\pi-x)=-\sec x$ and $\cos(\pi-x)=-\cos x$. Thus

$$\frac{\tan(\pi-x)}{\left[\sec(\pi-x)+\cos(\pi-x)\right]}=\frac{-\tan x}{-\sec x-\cos x}=\frac{\tan x}{\sec x+\cos x}$$

Now using property 6, that is, $\int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx,\ \text{if}\ f(2a-x)=f(x)$

$$2I=\pi\int_0^{\pi}\frac{\tan x}{(\sec x+\cos x)}dx=2\pi\int_0^{\pi/2}\frac{\tan x}{(\sec x+\cos x)}dx \quad (1)$$

Now, let $I_1 = \int_0^{\pi/2} \frac{\tan x}{(\sec x + \cos x)} dx$. Using **P-4**, we get:

$$I_1 = \int_0^{\pi/2} \frac{\tan\left(\frac{\pi}{2} - x\right)}{\sec\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\pi/2} \frac{\cot x}{\operatorname{cosec} x + \sin x} dx = \int_0^{\pi/2} \frac{\cos x}{\sin x(\operatorname{cosec} x + \sin x)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx.$$

Putting $z = \sin x$ ➔ $dz = \cos x\, dx$.

Also if $x = 0$ then $z = \sin 0 = 0$ and if $x = \pi/2$ then $z = \sin \pi/2 = 1$. Thus,

Thus, $2I = 2\pi\frac{\pi}{4} = \frac{\pi^2}{2}$ ➔ $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

(vii) $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$. **Using P-4, we get**

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx. \text{ Thus,}$$

$$I + I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

Thus, $2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$ ➔ $I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$ (1)

Putting $z = \tan x/2$ ➔ $dx = 2dz/(1 + z^2)$, $\sin x = 2z\, dz/(1 + z^2)$ and $\cos x = (1 - z^2)/(1 + z^2)$. [See section 1.7 Chapter 1]. Moreover,

If $x = 0$ then $z = \tan 0 = 0$ and if $x = \pi/2$ then $z = \tan \pi/4 = 1$.Thus equation (1) becomes,

$$I = \frac{1}{2}\int_0^1 \frac{1}{\sin x + \cos x} dx = \frac{1}{2}\int_0^1 \frac{1 + z^2}{2z + 1 - z^2} \cdot \frac{2dz}{1 + z^2} = \int_0^1 \frac{1}{1 - \left(z^2 - 2z\right)} dz$$

NOTE: $\sin x + \cos x = \frac{2z}{1 + z^2} + \frac{1 - z^2}{1 + z^2} = \frac{2z + 1 - z^2}{1 + z^2} \Rightarrow \frac{1}{\sin x + \cos x} = \frac{1 + z^2}{2z + 1 - z^2}$

Thus, $I = \int_0^1 \frac{1}{1 - \left(z^2 - 2z + 1 - 1\right)} dz = \int_0^1 \frac{1}{1 - \left(z^2 - 2z + 1\right) + 1} dz = \int_0^1 \frac{1}{2 - (z - 1)^2} dz$

$$= \int_0^1 \frac{1}{\left(\sqrt{2}\right)^2 - (z - 1)^2} dz \quad (2)$$

Substituting $z - 1 = u$ ➔ $dz = du$. Moreover, if $z = 0$ then $u = -1$ and if $z = 1$ then $u = 0$. Thus (2) becomes:

$$I=\int_{-1}^{0}\frac{1}{\left(\sqrt{2}\right)^2-u^2}du=\frac{1}{2\sqrt{2}}\ln\left[\frac{\sqrt{2}+u}{\sqrt{2}-u}\right]_{-1}^{0}$$ **NOTE:** $\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\frac{a+x}{a-x}$

$$=\frac{1}{2\sqrt{2}}\ln\left[\frac{\sqrt{2}+u}{\sqrt{2}-u}\right]_{-1}^{0}=\frac{1}{2\sqrt{2}}\left[\ln\frac{\sqrt{2}+0}{\sqrt{2}-0}-\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]=\frac{1}{2\sqrt{2}}\left[\ln 1-\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right]$$

$$=\frac{1}{2\sqrt{2}}\left[0-\ln\frac{(2-1)}{\left(\sqrt{2}+1\right)^2}\right]=-\frac{1}{2\sqrt{2}}\ln\frac{1}{\left(\sqrt{2}+1\right)^2}=-\frac{1}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right)^{-2}$$

$$=-\frac{1}{2\sqrt{2}}\times -2\ln\left(\sqrt{2}+1\right)$$

Thus, $I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$

(viii) $\int_0^{\pi}\frac{x}{1+\sin x}dx$

Solution: Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx$. **Using P-4, we get**

$$I=\int_0^{\pi}\frac{x}{1+\sin x}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin x}dx=\int_0^{\pi}\frac{\pi}{1+\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin x}dx$$

$$I=\int_0^{\pi}\frac{\pi}{1+\sin x}dx-I \quad\Rightarrow 2I=\int_0^{\pi}\frac{\pi}{1+\sin x}dx \Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\sin x}dx \qquad (1)$$

Now, let $I_1=\int_0^{\pi}\frac{1}{1+\sin x}dx$

Putting $z=\tan x/2$ ➔ $dx=2dz/(1+z^2)$ and $\sin x=2z/(1+z^2)$.
Also if $x=0$ then $z=\tan 0=0$ and if $x=\pi$ then $z=\tan\pi/2=\infty$. Thus,

$$I_1=\int_0^{\pi}\frac{1}{1+\sin x}dx=\int_0^{\infty}\frac{1+z^2}{1+z^2+2z}\cdot\frac{2dz}{1+z^2}=2\int_0^{\infty}\frac{1}{(z+1)^2}dz=2\int_0^{\infty}(z+1)^{-2}dz=2\left[\frac{(z+1)^{-1}}{-1}\right]_0^{\infty}$$

$$=-2\left[\frac{1}{(z+1)}\right]_0^{\infty}=-2\left[\frac{1}{\infty}-\frac{1}{1}\right]=-2[0-1]=2$$. Thus equation (1) becomes:

$$I=\int_0^{\pi}\frac{x}{1+\sin x}dx=2.\frac{\pi}{2}=\pi$$

(ix) $\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2dx$

Solution: Let $I=\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2dx$

$I=\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2dx=\int_0^{\pi/2}x^2\csc^2 x\,dx$. Integrating by parts, we get:

$$I=\left[x^2(-\cot x)\right]_0^{\pi/2}-\int_0^{\pi/2}2x.(-\cot x)\,dx=-[0-0]+2\int_0^{\pi/2}x\cot x\,dx$$. NOTE: $\cot(\pi/2)=0$

Integrating by parts again and notice that: $\int\cot x\,dx=\ln(\sin x)$, we get

$$I=2[x\ln(\sin x)]_0^{\pi/2}-2\int_0^{\pi/2}1.\ln(\sin x)\,dx=2\left[\frac{\pi}{2}\ln\sin\left(\frac{\pi}{2}\right)-0\right]-2\int_0^{\pi/2}\ln(\sin x)\,dx$$

Now, $\ln[\sin(\pi/2)=\ln(1)=0$.

Also we have proved that $\int_0^{\pi/2}\ln(\sin x)\,dx=-\frac{\pi}{2}\ln 2$ [See Example 7 part (ii)]

Thus, $I=\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2dx=-2\left[\frac{\pi}{2}\ln 2\right]=\pi\ln 2$

(x) $\int_0^{\pi/2}\ln(\tan x+\cot x)\,dx$

Solution: $I=\int_0^{\pi/2}\ln(\tan x+\cot x)\,dx=\int_0^{\pi/2}\ln\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)dx=\int_0^{\pi/2}\ln\left(\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}\right)dx$

$$=\int_0^{\pi/2}\ln\left(\frac{1}{\cos x\sin x}\right)dx=\int_0^{\pi/2}[\ln 1-\ln(\cos x\sin x)]dx=\int_0^{\pi/2}[0-\ln(\cos x)-\ln(\sin x)]dx$$

$$=-\int_0^{\pi/2}\ln(\cos x)dx-\int_0^{\pi/2}\ln(\sin x)\,dx=-\int_0^{\pi/2}\ln\left[\cos\left(\frac{\pi}{2}-x\right)\right]-\int_0^{\pi/2}\ln(\sin x)\,dx\,.\quad \textbf{(P-4)}$$

$$=-\int_0^{\pi/2}\ln[\sin x]-\int_0^{\pi/2}\ln(\sin x)\,dx=-2\int_0^{\pi/2}\ln(\sin x)\,dx=-2\left(-\frac{\pi}{2}\ln 2\right)=\pi\ln 2\quad \textbf{[Ex. 7 (ii)]}$$

Thus, $\int_0^{\pi/2}\ln(\tan x+\cot x)\,dx=\pi\ln 2$

(xi) $\int_0^{\pi}x\ln(\sin x)\,dx$

Solution: Let $I=\int_0^{\pi}x\ln[\sin x]\,dx$ Using **P-4**, we get

$$I=\int_0^{\pi}(\pi-x)\ln[\sin(\pi-x)]\,dx=\int_0^{\pi}(\pi-x)\ln[\sin x]\,dx=\pi\int_0^{\pi}\ln[\sin x]\,dx-\int_0^{\pi}x\ln[\sin x]\,dx$$

$$I=\pi\int_0^{\pi}\ln[\sin x]\,dx-I\quad\Rightarrow 2I=\pi\int_0^{\pi}\ln[\sin x]\,dx\quad\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\ln[\sin x]\,dx\qquad(1)$$

Let $f(x)=\ln[\sin x]$ ➔ $f(\pi-x)=\ln[\sin(\pi-x)]=\ln[\sin x]=f(x)$.

Thus using **P-6**, we get

$$I=\frac{\pi}{2}2\int_0^{\pi/2}\ln[\sin x]\,dx=\pi\int_0^{\pi/2}\ln[\sin x]\,dx=\pi\left(-\frac{\pi}{2}\ln 2\right)=-\frac{\pi^2}{2}\ln 2.$$ **[See Example 7 (ii)]**

(xii) $\int_0^1\frac{\ln(1+x)}{1+x^2}dx$

Solution: Putting $x = \tan\theta$ ➔ $dx = \sec^2\theta\, d\theta$.
Also if $x = 0$ then $\tan\theta = 0$ ➔ $\theta = 0$ and if $x = 1$ then $\tan\theta = 1$ ➔ $\theta = \pi/4$. Thus,

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \ln\left[1+\tan(\pi/4 - \theta)\right] d\theta \qquad \textbf{[By P-4]}$$

$$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan\theta}\right] d\theta. \text{ Thus,}$$

$$I = \ln 2\int_0^{\pi/4} 1\, d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \ln 2[\theta]_0^{\pi/4}$$

➔ $2I = (\pi/4).\ \ln 2$ ➔ $I = (\pi/8).\ \ln 2$

(xiii) $\displaystyle\int_0^{\pi/2} \sin x \ln(\sin x)\, dx$

Solution: Let $I = \displaystyle\int_0^{\pi/2} \sin x \ln(\sin x)\, dx = \int_0^{\pi/2} (\sin x)\ln\left[\sqrt{1-\cos^2 x}\right] dx$

Substituting $z = \cos x$ ➔ $-dz = \sin x\, dx$.
When $x = 0$ then $z = \cos 0 = 1$ and when $x = \pi/2$ then $z = \cos \pi/2 = 0$. Thus,

$$I = -\int_1^0 \ln\sqrt{1-z^2}\, dz = \int_0^1 \ln\left[\sqrt{1-z^2}\right] dz = \frac{1}{2}\int_0^1 \ln(1+z^2)\, dz$$

Now Maclaurin's series for $\ln\left[1-z^2\right] = -\frac{1}{2}\left[z^2+\frac{z^4}{2}+\frac{z^6}{3}+\ldots\right]$. Thus,

$$I = -\frac{1}{2}\int_0^1\left[z^2+\frac{z^4}{2}+\frac{z^6}{3}+\ldots\right]dz = -\frac{1}{2}\left[\frac{z^3}{3}+\frac{z^5}{2.5}+\frac{z^7}{3.7}+\ldots\right]_0^1 = -\left[\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\ldots\right]$$

$$= -\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)\ldots\right] = \left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right] \qquad \textbf{[Add and subtract 1]}$$

$$= \left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right] - 1 = \ln 2 - 1 = \ln 2 - \ln e = \ln\left(\frac{2}{e}\right)$$

REMARK: $\ln e = 1$ and Maclaurin's series for $\ln 2 = \left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]$

(xiv) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx$

Solution: Let $I = \displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx$. **Using P-4, we get**

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)+\cos(\pi/2 - x)}dx = \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}dx$$

Thus, $I + I = \displaystyle\int_0^{\pi/2} \frac{\sin x+\cos x}{\sin x+\cos x}dx = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2I = \frac{\pi}{2}$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}dx=\frac{\pi}{4}$$

(xv) $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

Solution: Let $I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$. **Using P-4, we get**

$$I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx-\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$$

$$I=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx-I \quad\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx=\pi\int_0^{\pi}\frac{1+\sin x-1}{1+\sin x}dx$$

$$2I=\pi\int_0^{\pi}\left[\frac{1+\sin x}{1+\sin x}-\frac{1}{1+\sin x}\right]dx=\pi\int_0^{\pi}\left[1-\frac{1}{1+\sin x}\right]dx=\pi\int_0^{\pi}1dx-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$=\pi[\pi-0]-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx=\pi^2-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx \qquad (1)$$

Putting $z=\tan x/2$ $\Rightarrow$ $dx=2dz/(1+z^2)$ and $\sin x=2z/(1+z^2)$.
Also if $x=0$ then $z=\tan 0=0$ and if $x=\pi$ then $z=\tan\pi/2=\infty$. Thus,

$$\int_0^{\pi}\frac{1}{1+\sin x}dx=\int_0^{\infty}\frac{1+z^2}{1+z^2+2z}\cdot\frac{2dz}{1+z^2}=2\int_0^{\infty}\frac{1}{(z+1)^2}dz=2\int_0^{\infty}(z+1)^{-2}dz=2\left[\frac{(z+1)^{-1}}{-1}\right]_0^{\infty}$$

$$=-2\left[\frac{1}{(z+1)}\right]_0^{\infty}=-2\left[\frac{1}{\infty}-\frac{1}{1}\right]=-2[0-1]=2$$. Thus equation (1) becomes:

$$2I=\pi^2-\pi(2)\quad\Rightarrow I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx=\frac{\pi^2}{2}-\pi$$

(xvi) $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$

Solution: Let $I=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$. **Using P-4, we get**

$$I=\int_0^{\pi/2}\frac{\sin(\pi/2-x)-\cos(\pi/2-x)}{1+\sin(\pi/2-x)\cos(\pi/2-x)}dx=\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\cos x\sin x}dx$$

Thus $I+I=\int_0^{\pi/2}\frac{\sin x-\cos x+\cos x-\sin x}{1+\sin x\cos x}dx=0 \quad\Rightarrow 2I=0$

$$\Rightarrow\ I=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx=0$$

(xvii) $\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin x\cos x}dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x} dx$. **Using P-4, we get**

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{1+\sin(\pi/2 - x)\cos(\pi/2 - x)} dx = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos x \sin x} dx$$

$$\text{Thus, } I + I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1+\sin x \cos x} dx = \int_0^{\pi/2} \frac{1}{1+\cos x \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{1}{1+\sin x \cos x} dx \qquad (1)$$

Now substituting: $z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\Rightarrow dx = dz/\sec^2 x = dz/(1 + \tan^2 x) = dz/(1 + z^2)$.

NOTE: $\tan x = z = z/1 = P/B \Rightarrow B = 1$ and $P = z \Rightarrow H^2 = B^2 + P^2 = 1 + z^2$

Thus, $\sin x \cos x = z/(1 + z^2)$. Also if $x = 0$ then $z = \tan 0 = 0$ and when $x = \pi/2$ then $z = \tan \pi/2 = \infty$. Hence, equation (1) becomes:

$$2I = \int_0^{\infty} \frac{1}{1+z/(1+z^2)} \cdot \frac{dz}{(1+z^2)} = \int_0^{\infty} \frac{1+z^2}{1+z^2+z} \cdot \frac{dz}{(1+z^2)} = \int_0^{\infty} \frac{1}{z^2+z+1} dz$$

$$= \int_0^{\infty} \frac{1}{[z^2+z+1/4+3/4]} dz = \int_0^{\infty} \frac{1}{[z^2+z+(1/4)]+(3/4)} dz = \int_0^{\infty} \frac{1}{[z+(1/2)]^2+(\sqrt{3}/2)^2} dz$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{(z+1/2)}{\sqrt{3}/2} \right]_0^{\infty} \qquad \textbf{NOTE: } \int_0^{\infty} \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1} \infty - \tan^{-1}(1/\sqrt{3}) \right] = \frac{2}{\sqrt{3}} \left[\frac{\pi}{2} - \frac{\pi}{6} \right] = \frac{2}{\sqrt{3}} \left[\frac{2\pi}{6} \right] = \frac{2\pi}{3\sqrt{3}} \qquad \Rightarrow I = \frac{\pi}{3\sqrt{3}}$$

$$\text{Thus } \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x} dx = \frac{\pi}{3\sqrt{3}}$$

9.2 BETA AND GAMMA FUNCTIONS

In this section we study two important types of integrals that have many applications in applied sciences and engineering. They are (i) Beta Function (ii) Gamma Function

Definition: If m and n are positive numbers, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is called **Beta function** of first kind and is denoted by $\beta(m,n)$ (read as "Beta m, n"). Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m > 0, n > 0$$

Beta function is also called Eulerian integral of first kind (after Euler, a great mathematician).

Properties of Beta Function

(i) Symmetry of the Beta Function: Prove that $\beta(m,n) = \beta(n,m)$

Proof: By definition, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m > 0, n > 0$

Changing x by 1 − x, we get

$$\beta(m,n)=\int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}dx=\int_0^1 (1-x)^{m-1}(x)^{n-1}dx=\int_0^1 x^{n-1}(1-x)^{m-1}dx=\beta(n,m)$$

Hence, $\beta(m,n)=\beta(n,m)$.

(ii) If both m and n are positive integers, then $\beta(m,n)=\dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

By definition, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx,\quad m>0, n>0$.

Integrating by parts taking $u = x^{m-1}$ and $v = (1-x)^{n-1}$, we get

$$\beta(m,n)=\left[x^{m-1}\times\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1\frac{(1-x)^n}{-n}(m-1)x^{m-2}dx=(0-0)+\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n dx$$

$$=\frac{m-1}{n}\int_0^1 x^{(m-1)-1}(1-x)^{(n+1)-1}dx=\frac{m-1}{n}\beta(m-1,n+1) \qquad (1)$$

Changing m to m − 1 and n to n + 1 in (1), we have

$$\beta(m-1,n+1)=\frac{m-2}{n+1}\beta(m-2,n+2)$$

Substituting this into (1) we get

$$\beta(m,n)=\frac{m-1}{n}\frac{m-2}{n+1}\beta(m-2,n+2) \qquad (2)$$

Repeating the above process, we get

$$\beta(m,n)=\frac{(m-1)(m-2)\ldots\{m-(m-1)\}}{n(n+1)\ldots\{n+(m-2)\}}\times\beta\{m-(m-1),n+(m-1)\}$$

$$\beta(m,n)=\frac{(m-1)(m-2)\ldots1}{n(n+1)\ldots(n+m-2)}\beta(1,n+m-1) \qquad (3)$$

But, $\beta(1,n+m-1)=\int_0^1 x^0(1-x)^{n+m-2}dx=\int_0^1(1-x)^{n+m-2}dx$

$$=-\frac{1}{n+m-1}\left[(1-x)^{n+m-2}\right]_0^1=-\frac{1}{n+m-1}(0-1)=\frac{1}{n+m-1}$$

Thus from (3), we get

$$\beta(m,n)=\frac{(m-1)(m-2)\ldots1}{n(n+1)\ldots(n+m-2)(n+m-1)}=\frac{(m-1)!}{n(n+1)\ldots(n+m-2)(n+m-1)} \qquad (4)$$

NOTE: Denominator is written in reversed order.

Multiplying (4) by (n − 1)!, we get

$$\beta(m,n)=\frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2)\ldots(n+1)n(n-1)!}=\frac{(m-1)!(n-1)!}{(n+m-1)!}$$

Corollary: $\beta(1,n)=1/n$

Substituting $m=1$ in above, we get

$$\beta(1,n)=\frac{(1-1)!(n-1)!}{(n+1-1)!}=\frac{(n-1)!}{n!}=\frac{(n-1)!}{n(n-1)!}=\frac{1}{n}$$

Example 01: Evaluate: $\int_0^2 \left(8-x^3\right)^{-1/3}dx$

Solution: Let $x^3 = 8z$ ➔ $x = 2z^{1/3}$ ➔ $dx = 2/3\ x^{-2/3}\ dz$.
Also if $x = 0$ the $z = 0$ and if $x = 2$ then $z = 1$. Therefore,

$$\int_0^2 \left(8-x^3\right)^{-1/3}dx = \int_0^1 (8-8z)^{-1/3}\frac{2}{3}z^{-2/3}dz = \frac{2}{3}\int_0^1 8^{-1/3}(1-z)^{-1/3}z^{-2/3}dz$$

$$= \frac{2}{3}\times\frac{1}{2}\int_0^1 z^{-2/3}(1-z)^{-1/3}dz = \frac{1}{3}\int_0^1 z^{\frac{1}{3}-1}(1-z)^{\frac{2}{3}-1}dz = \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right).$$

Example 02: Evaluate: $\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx$

Solution: Let $\sqrt{x} = z$ ➔ $x = z^2$ ➔ $dx = 2z\ dz$
Also if $x = 0$ the $z = 0$ and if $x = 1$ then $z = 1$. Therefore,

$$\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx = \int_0^1 \left(z^2\right)^2(1-z)^5(2z)dz = 2\int_0^1 t^9(1-z)^5 dz$$

$$= 2\beta(10,6) = 2\frac{(10-1)!\,(6-1)!}{(10+6-1)!} = 2\frac{9!\,5!}{15!} = \frac{1}{15015}$$

Example 03: Evaluate: $\int_0^1 \left(1-x^3\right)^{-1/2}dx$

Solution: Let $x^3 = z$ ➔ $x = z^{1/3}$ ➔ $dx = 1/3\ (z)^{-2/3}\ dz$
Also if $x = 0$ the $z = 0$ and if $x = 1$ then $z = 1$. Therefore,

$$\int_0^1 \left(1-x^3\right)^{-1/2}dx = \int_0^1 (1-z)^{-1/2}\left(z^{-2/3}/3\right)dz = \frac{1}{3}\int_0^1 z^{-2/3}(1-z)^{-1/2}dz = \frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right)$$

Two Other Forms of Beta Function

(I) We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ \ m>0, n>0$.

Substituting $x = \sin^2\theta$ ➔ $dx = 2\sin\theta\cos\theta\, d\theta$.
Also when $x = 0$ then ➔ $\sin\theta = 0$ ➔ $\theta = 0$.
When $x = 1$ then ➔ $\sin\theta = 1$ ➔ $\theta = \pi/2$.

Thus $$\beta(m,n) = \int_0^{\pi/2}\left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1}2\sin\theta\cos\theta\, d\theta$$

➔ $$\beta(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

(II) By definition, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ \ m>0, n>0$.

Putting: $$x = \frac{1}{1+y} \Rightarrow 1-x = 1-\frac{1}{1+y} = \frac{1+y-1}{1+y} = \frac{y}{1+y}$$
Also $1 + y = 1/x$ ➔ $y = (1/x) - 1$. If $x = 0$ then $y = \infty$ and if $x = 1$ then $y = 0$.
Finally, $dx = -dy/(1+y)^2$. Thus,

$$\beta(m,n) = -\int_{\infty}^{0} \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy = \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Definition: If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the **Gamma function** and is denoted by $\Gamma(n)$ (read as "Gamma n"). Thus,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad \text{where } n > 0 \cdot$$

Gamma function is also called Eulerian integral of second kind or it is also called the "Generalized factorial function."

By definition, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.

Putting n = 1, we get:

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 dx = \lim_{M\to\infty} \int_0^M e^{-x}\, dx = -\lim_{M\to\infty} e^{-x}\Big|_0^M = -\lim_{M\to\infty}\left(e^{-M} - e^0\right) = 1 \cdot$$

Now integrating by parts, we get

$$\Gamma(n) = -x^{n-1}e^{-x}\Big|_0^\infty - \int_0^\infty (n-1)x^{n-2}\left(-e^{-x}\right) dx = -\lim_{M\to\infty} \frac{x^{n-1}}{e^x}\Big|_0^M + (n-1)\int_0^\infty e^{-x} x^{(n-1)-1} dx$$

$$= -\lim_{M\to\infty} \frac{M^{n-1}}{e^M} - 0 + (n-1)\Gamma(n-1) = (n-1)\Gamma(n-1)$$

Thus, $\Gamma(n) = (n-1)\Gamma(n-1)$

By repeated application of this formula, we get

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)(n-3)\Gamma(n-3)\ldots$$

$$= (n-1)(n-2)(n-3)\ldots 3.2.1\ \Gamma(1)$$

$\therefore \quad \Gamma(n) = (n-1)!$ since, $\Gamma(1) = 1$ (1)

This formula is valid only when n is a positive integer. For example, $\Gamma(5) = 4! = 24$. In case n is positive but not an integer, we use the recurrence formula

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

For example, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

REMARK: $\Gamma(1/2) = \sqrt{\pi}$ [See below]

In case, if n is negative integer, we may evaluate the gamma function as follows:

$\Gamma(n) = (n-1)\Gamma(n-1) \quad \Rightarrow \Gamma(n+1) = n\Gamma(n) \quad \Rightarrow \Gamma(n) = \Gamma(n+1)/n$. For example,

$$\Gamma(-4) = \Gamma(-4+1)/(-4) = \Gamma(-3)/(-4)$$

$$= \Gamma(-3+1)/\left[(-3)(-4)\right] = \Gamma(-2)/12$$

$$= \Gamma(-2+1)/(-2)(12) = \Gamma(-1)/(-24)$$

$$= \Gamma(0)/(-1+1)(-24) = \Gamma(0)/(0)(-24) = \infty$$

Also notice that: $\Gamma(0)=\Gamma(1)/0=\infty$

This shows that if n is zero or a negative integer $\Gamma(n)$ is infinite. However, if n is not a negative integer, we use the formula $\Gamma(n-1)=\Gamma(n)/(n-1)$ to evaluate the value of $\Gamma(n)$.

For example,

$$\Gamma\left(\frac{-7}{2}\right)=\frac{\Gamma(-5/2)}{(-7/2)}=\frac{\Gamma(-3/2)}{(-5/2)(-7/2)}=\frac{\Gamma(-1/2)}{(-3/2)(-5/2)(-7/2)}$$

$$=\frac{\Gamma(1/2)}{(-1/2)(-3/2)(-5/2)(-7/2)}=\frac{2^4}{1.3.5.7}\Gamma\left(\frac{1}{2}\right)=\frac{16}{105}\sqrt{\pi}$$

Example 04: Evaluate $\int_0^\infty x^{1/4}e^{-\sqrt{x}}dx$

Solution: Putting $z=\sqrt{x}$ ➔ $z^2=x$ ➔ $2\,z\,dz=dx$. Thus

$$\int_0^\infty x^{1/4}e^{-\sqrt{x}}\,dx=\int_0^\infty\left(z^2\right)^{1/4}e^{-z}2z\,dz=2\int_0^\infty z^{3/2}e^{-z}\,dz=2\,\Gamma\left(\frac{5}{2}\right)$$ **[By definition of Gamma Function]**

$$=2\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=3.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{2}\sqrt{\pi}$$ [Note: $\Gamma(n)=(n-1)\Gamma(n-1)$]

Example 05: Evaluate $\int_0^\infty \sqrt{x}\;e^{-\sqrt[3]{x}}dx$

Solution: Putting $z=\sqrt[3]{x}$ ➔ $z^3=x$ ➔ $3\,z^2\,dz=dx$. Thus

$$\int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}\,dx=\int_0^\infty\left(z^3\right)^{1/2}e^{-z}3z^2\,dz=3\int_0^\infty z^{7/2}e^{-z}dz=3\,\Gamma\left(\frac{9}{2}\right)$$ **[By definition of Gamma Function]**

$$=3\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{315}{16}\sqrt{\pi}$$ [Note: $\Gamma(n)=(n-1)\Gamma(n-1)$]

Example 06: Evaluate $\int_0^\infty x^{n-1}e^{-k^2x^2}dx$

Solution: Putting $z=k^2x^2$ ➔ $x=\sqrt{z}/k$ ➔ $dx=dz/2k\sqrt{z}$. Thus

$$I=\int_0^\infty\left(\frac{\sqrt{z}}{k}\right)^{n-1}e^{-z}\frac{dz}{2k\sqrt{z}}=\frac{1}{2k.k^{n-1}}\int_0^\infty z^{(n-1)/2}.z^{-1/2}e^{-z}dz=\frac{1}{2k^n}\int_0^\infty z^{\frac{n}{2}-1}e^{-z}\,dz=\frac{1}{2k^n}\,\Gamma\left(\frac{n}{2}\right)$$

Example 07: Evaluate $\int_0^\infty \frac{x^h}{h^x}dx$

Solution: Putting $h^x=e^z$ ➔ $\ln(h^x)=\ln(e^z)$ ➔ $x\ln h=z$

➔ $x=z/\ln h$ ➔ $dx=dz/\ln h$. Thus

$$I=\int_0^\infty\frac{x^h}{h^x}dx=\int_0^\infty\left(\frac{z}{\ln h}\right)^h.\frac{1}{e^z\ln h}dz=\frac{1}{(\ln h)^{h+1}}\int_0^\infty e^{-z}z^h\,dz=\frac{1}{(\ln h)^{h+1}}\Gamma(h+1)$$

Relation between Beta and Gamma Functions

We know that, $\beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$. But $\Gamma(m)=(m-1)!$.

Using this formula, we get: $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Example 08: Prove that $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\dfrac{1}{2}\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

Solution: We know that $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Putting $2m-1=p$ and $2n-1=q$ ➔ $m=(p+1)/2$ and $n=(q+1)/2$.

Thus, $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\dfrac{1}{2}\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

Example 09: Prove that: $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Proof: We know that, $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ Taking $m=n=1/2$, we get

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)}=\frac{[\Gamma(1/2)]^2}{\Gamma(1)}=[\Gamma(1/2)]^2 \qquad [\because \Gamma(1)=1]$$

Now, $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

Thus, $\beta\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\pi/2}\sin^0\theta\cos^0\theta\,d\theta=2\int_0^{\pi/2}1\,d\theta=2[\theta]_0^{\pi/2}=2\frac{\pi}{2}=\pi$

Hence, $\beta\left(\frac{1}{2},\frac{1}{2}\right)=\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\pi \;\Rightarrow\; \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\pi\Rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Example 10: Prove that: $\int_0^{\infty}e^{-x^2}dx=\dfrac{1}{2}\sqrt{\pi}$

Proof: Let

$z=x^2\Rightarrow dz=2x\,dx\Rightarrow dx=\dfrac{1}{2\sqrt{z}}dz$. Also if $x=0\Rightarrow z=0$ and if $x\to\infty\Rightarrow z\to\infty$

Therefore, $\int_0^{\infty}e^{-x^2}dx=\frac{1}{2}\int_0^{\infty}e^{-z}\frac{1}{\sqrt{z}}dz=\frac{1}{2}\int_0^{\infty}e^{-z}(z)^{-1/2}dz=\frac{1}{2}\int_0^{\infty}e^{-z}(z)^{\frac{1}{2}-1}dz=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}$

Example 11: Prove that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\dfrac{\pi}{\sqrt{2}}$

Proof: Let $I=\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\int_0^{\pi/2}\sqrt{\dfrac{\sin\theta}{\cos\theta}}\,d\theta=\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$

Using relation between Beta and Gamma function, that is,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \Rightarrow \frac{1}{2}\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Taking p = 1/2 and q = - (1/2) , we get

$$I = \frac{1}{2}\frac{\Gamma\left(\frac{1/2+1}{2}\right)\Gamma\left(\frac{-1/2+1}{2}\right)}{\Gamma\left(\frac{1/2+1}{2}+\frac{-1/2+1}{2}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$I = \frac{1}{2}\frac{\sqrt{2}\pi}{1} = \pi/\sqrt{2}.$$

NOTE : $\Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi$

REMARK: $\Gamma(1-n).\Gamma(n) = \pi/\sin n\pi$. Puuting $n = 1/4$, we get :

$$\Gamma(3/4).\Gamma(1/4) = \pi/(\sin\pi/4) = \pi/\left(1/\sqrt{2}\right) = \sqrt{2}\pi$$

9.3 IMPROPER INTEGRALS

Recall that definite integration of a function y = f(x) between x = a and x = b is given by:

$$\int_a^b f(x)\, dx \text{ , provided integral exists.}$$

Here the interval between a and b can be extended infinitely far in either direction. The three possible types of intervals are shown next. The corresponding integrals are called "Improper Integrals."

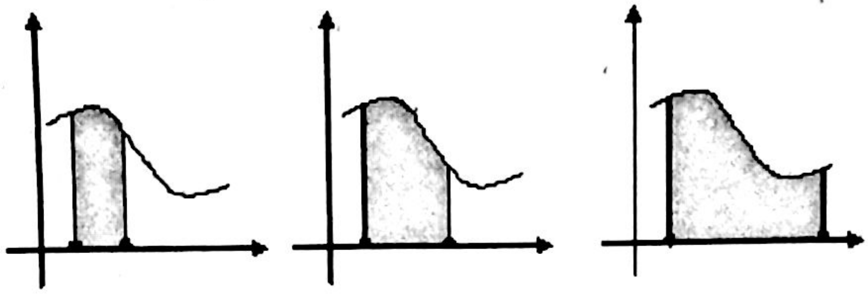
Interval	Integral
$[a,\infty)$	$\int_a^\infty f(x)dx$
$(-\infty,b]$	$\int_{-\infty}^b f(x)dx$
$(-\infty,\infty)$	$\int_{-\infty}^\infty f(x)dx$

An integral with finite limits a, b (say) can also be considered as an "Improper Integral" if the integrand f(x) becomes infinite within the finite interval of integration. For example,

$$\int_0^2 \frac{1}{x-1}dx \text{ and } \int_0^3 \frac{1}{x-3}dx$$

are improper integrals.

In order to motivate a definition for the improper integral $\int_a^\infty f(x)\, dx$, let us consider the following definite integral: $\int_a^b f(x)\, dx$. The following figure shows the area under the curve f(x) for larger and larger values of b.



The area under the curve for larger and larger values of b

Thus, the improper integral from a to ∞ can be defined as:

$$\int_a^{\infty} f(x)\,dx = \lim_{m\to\infty}\int_a^{m} f(x)\,dx,\ \forall m > a$$

provided f(x) is continuous on the interval $[a,\infty)$.

Example 01: Evaluate the following improper integrals

$$(i)\ \int_1^{\infty}\sqrt{x}\,dx \quad (ii)\ \int_1^{\infty}\frac{1}{x^2}\,dx$$

Solution: (i) $\int_1^{\infty}\sqrt{x}\,dx = \lim_{b\to\infty}\int_1^{b}(x)^{1/2}\,dx = \lim_{b\to\infty}\left[\frac{x^{3/2}}{3/2}\right]_1^b = \frac{2}{3}\lim_{b\to\infty}\left(b^{3/2}-1^{3/2}\right) = \frac{2}{3}\lim_{b\to\infty}\left(b^{3/2}-1\right)\to\infty$.

Hence given integral does not exist.

(ii) $\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b}(x)^{-2}\,dx = \lim_{b\to\infty}\left[\frac{x^{-1}}{-1}\right]_1^b = -\lim_{b\to\infty}\left(\frac{1}{b}-1\right) = -(0-1) = 1$

Note: The integral evaluated in part (i) is said to be divergent, since it has no finite numerical value. By contrast, the integral in part (ii) is said to be convergent, since it has a finite value. There is indeed a finite area under the curve.

Improper integrals of the type $\int_{-\infty}^{b} f(x)\,dx$ are defined and evaluated in similar manner. That

is;
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{m\to-\infty}\int_{m}^{b} f(x)\,dx$$

provided f(x) is continuous on the interval $(-\infty, b]$.

Example 02: Evaluate $\int_{-\infty}^{-2}\frac{dx}{(4-x)^2}$

Solution: $\int_{-\infty}^{-2}\frac{dx}{(4-x)^2} = \lim_{a\to-\infty}\int_a^{-2}(4-x)^{-2}\,dx = \lim_{a\to-\infty}\left[-\frac{(4-x)^{-1}}{-1}\right]_a^{-2}$

$$= \lim_{a\to-\infty}\left[\frac{1}{4-x}\right]_a^{-2} = \lim_{a\to-\infty}\left(\frac{1}{6}-\frac{1}{4-a}\right) = \frac{1}{6}$$

Improper integrals of the form $\int_{-\infty}^{\infty} f(x)dx$ are defined and evaluated as follows and be evaluated accordingly.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

provided that integral exists. If either (or both) of the two integrals diverges, then given integral also diverges otherwise, it converges.

Example 03: Evaluate the following improper integrals

$$(i)\ \int_{-\infty}^{\infty} \frac{x}{x^4+1}dx \quad (ii)\ \int_{-\infty}^{\infty} xe^{-x^2}dx$$

Solution: (i) Consider $\int \frac{x}{x^4+1}dx$. Substituting $x^2 = z \Rightarrow 2xdx = dz \Rightarrow xdx = dz/2$.

Then $\int \frac{x}{x^4+1}dx = \int \frac{1/2\,dz}{z^2+1} = \frac{1}{2}\int \frac{1}{z^2+1}dz = \frac{1}{2}\tan^{-1} z = \frac{1}{2}\tan^{-1}(x^2)$.

Now,
$$= \int_{-\infty}^{0} \frac{x}{x^4+1}dx + \int_{0}^{\infty} \frac{x}{x^4+1}dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{x}{x^4+1}dx + \lim_{b\to\infty}\int_{0}^{b} \frac{x}{x^4+1}dx$$

$$= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}x^2\right]_a^0 + \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}x^2\right]_0^b$$

$$= \frac{1}{2}\lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a^2\right) + \frac{1}{2}\lim_{b\to\infty}\left(\tan^{-1}b^2 - \tan^{-1}0\right)$$

➔ $$\int_{-\infty}^{\infty} \frac{x}{x^4+1}dx = -\frac{1}{2}\lim_{a\to-\infty}\tan^{-1}a^2 + \frac{1}{2}\lim_{b\to\infty}\tan^{-1}b^2 = -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) = 0$$

(ii) Consider $\int xe^{-x^2}dx$. Substituting $x^2 = z \Rightarrow 2xdx = dz \Rightarrow xdx = \frac{1}{2}dz$. Then

$$\int xe^{-x^2}dx = \frac{1}{2}\int e^{-z}dz = \frac{1}{2}e^{-z}(-1) = \frac{-1}{2}e^{-x^2}$$

Now let, $I = \int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx = \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$

$$= \operatorname{Lim}_{a\to-\infty}\left(\frac{-1}{2}\right)\left[e^{-x^2}\right]_a^0 + \operatorname{Lim}_{b\to\infty}\left(\frac{-1}{2}\right)\left[e^{-x^2}\right]_0^b = \frac{-1}{2}\operatorname{Lim}_{a\to-\infty}\left(e^0 - e^{-a^2}\right) - \frac{1}{2}\operatorname{Lim}_{b\to\infty}\left(e^{-b^2} - e^0\right).$$

➔ $$I = -\frac{1}{2}(1-0) - \frac{1}{2}(0-1) = -\frac{1}{2} + \frac{1}{2} = 0$$

9.4 REDUCTION FORMULAE

Definition: Reduction formula is a formula that connects a given integral with another integral which is of the same type but of a lower degree or of a lower order.

We have already discussed such formulae in the previous Chapter however, we shall now discuss them in the light of "Definite Integration".

We know that: (i) $\int \sin^n x\,dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n}{n-1}\int \sin^{n-2} x\,dx$

And (ii) $\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n}{n-1}\int \cos^{n-2} x\,dx$

Now $\sin 0 = \sin \pi = \sin 2\pi = \cos \pi/2 = 0$. Thus, if the limits of integration are taken from 0 to $\pi/2$ or 0 to π or 0 to 2π, the first term on right side becomes zero. In this case, we have:

(a) $\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$ (b) $\int_0^{\pi} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi} \sin^{n-2} x\, dx$

(c) $\int_0^{2\pi} \sin^n x\, dx = \frac{n-1}{n}\int_0^{2\pi} \sin^{n-2} x\, dx$ (d) $\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\, dx$

(e) $\int_0^{\pi} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\pi} \cos^{n-2} x\, dx$ (f) $\int_0^{2\pi} \cos^n x\, dx = \frac{n-1}{n}\int_0^{2\pi} \cos^{n-2} x\, dx$

Example 01: Evaluate the following integrals using reduction formula.

(i) $\int_0^{2\pi} \sin^6 x\, dx$ (ii) $\int_0^{2\pi} \sin^7 x\, dx$ (iii) $\int_0^{\pi/2} \cos^7 x\, dx$

Solution: (i) $\int_0^{2\pi} \sin^6 x\, dx = \frac{6-1}{6}\int_0^{2\pi} \sin^4 x\, dx = \frac{5}{6}\cdot\frac{4-1}{4}\int_0^{2\pi} \sin^2 x\, dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{2-1}{2}\int_0^{2\pi} \sin^0 x\, dx$

Thus, $\int_0^{2\pi} \sin^6 x\, dx = \frac{5}{16}\int_0^{2\pi} 1\, dx = \frac{5}{16}[x]_0^{2\pi} = \frac{5}{16}(2\pi) = \frac{5\pi}{8}$

(ii) $\int_0^{2\pi} \sin^7 x\, dx = \frac{6}{7}\int_0^{2\pi} \sin^5 x\, dx = \frac{6}{7}\cdot\frac{4}{5}\int_0^{2\pi} \sin^3 x\, dx = \frac{24}{35}\cdot\frac{2}{3}\int_0^{2\pi} \sin^1 x\, dx = \frac{16}{35}[-\cos x]_0^{2\pi}$

Thus, $\int_0^{2\pi} \sin^7 x\, dx = -\frac{16}{35}[\cos 2\pi - \cos 0] = -\frac{16}{35}[1-1] = 0$

(iii) $\int_0^{\pi/2} \cos^7 x\, dx = \frac{6}{7}\int_0^{\pi/2} \cos^5 x\, dx = \frac{6}{7}\cdot\frac{4}{5}\int_0^{\pi/2} \cos^3 x\, dx = \frac{24}{35}\cdot\frac{2}{3}\int_0^{\pi/2} \cos^1 x\, dx = \frac{16}{35}[\sin x]_0^{\pi/2}$

Thus, $\int_0^{\pi/2} \cos^7 x\, dx = \frac{16}{35}[\sin(\pi/2) - \sin 0] = \frac{16}{35}[1-0] = \frac{16}{35}$

Reduction Formula for $\int \sin^p x \cos^q x\, dx$

The integral $\int \sin^p x \cos^q x\, dx$ can be connected with any one of the following six integrals to get a reduction formula for it.

(i) $\int \sin^{p-2} x \cos^q x\, dx$ (ii) $\int \sin^p x \cos^{q-2} x\, dx$ (iii) $\int \sin^{p+2} x \cos^q x\, dx$

(iv) $\int \sin^p x \cos^{q+2} x\, dx$ (v) $\int \sin^{p-2} x \cos^{q+2} x\, dx$ (vi) $\int \sin^{p+2} x \cos^{q-2} x\, dx$

Thus, in finding a reduction formula for $\int \sin^p x \cos^q x\, dx$, we may

- Decrease or increase by 2 the index of sin x leaving that of cos x unchanged as in (i) or (ii).
- Decrease or increase by 2 the index of cos x leaving that of sin x unchanged as in (iii) or (iv).
- Decrease the index of sin x by 2 and increase that of cos x by 2 as in (v).
- Increase the index of sin x by 2 and decrease that of cos x by 2 as in (vi).

But we cannot increase or decrease by 2 the indices of both sin x and cos x in the same formula.

Method for connecting $\int \sin^p x \cos^q x\, dx$ with any one of the above six integrals:

Step 1: Take $P=\sin^{\lambda+1}x\cos^{\mu+1}x$, where λ is smaller of the two indices of sin x and μ is smaller of the cos x in the two integrals which are to be connected.

Step 2: Find dP/dx and express it as a linear function of the two integrands whose integrals are being connected.

Step 3: Integrate both sides with respect to x, transpose and solve for the given integral.

Example 02: Connect $\int \sin^p x\cos^q x\,dx$ with $\int \sin^{p-2}x\cos^q x\,dx$. Hence evaluate $\int \sin^4 x\cos^2 x\,dx$.

Solution: Let $P=\sin^{p-2+1}x\cos^{q+1}x=\sin^{p-1}x\cos^{q+1}x$

Differentiating both sides with respect to x, we get

$$dP/dx=(p-1)\sin^{p-2}x(\cos x)\cos^{q+1}x+\sin^{p-1}x(q+1)\cos^q x(-\sin x)$$
$$=(p-1)\sin^{p-2}x\cos^{q+2}x-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x\cos^2 x-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x(1-\sin^2 x)-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x-(p-1)\sin^p x\cos^q x-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x-(p-1+q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x-(p+q)\sin^p x\cos^q x$$

Integrating both sides with respect to x, we get

$$P=(p-1)\int\sin^{p-2}x\cos^q x\,dx-(p+q)\int\sin^p x\cos^q x\,dx$$
$$(p+q)\int\sin^p x\cos^q x\,dx=-P+(p-1)\int\sin^{p-2}x\cos^q x\,dx$$

$$\int\sin^p x\cos^q x\,dx=-\frac{\sin^{p-1}x\cos^{q+1}x}{(p+q)}+\frac{(p-1)}{(p+q)}\int\sin^{p-2}x\cos^q x\,dx \qquad (1)$$

which is the required reduction formula. If the power of cos x is reduced, we get

$$\int\sin^p x\cos^q x\,dx=-\frac{\sin^{p-1}x\cos^{q+1}x}{(p+q)}+\frac{(q-1)}{(p+q)}\int\sin^p x\cos^{q-2}x\,dx \qquad (2)$$

For example, to evaluate $\int\sin^4 x\cos^2 x\,dx$, put p = 4 and q = 2 in (1), we get

$$\int\sin^4 x\cos^2 x\,dx=-\frac{\sin^3 x\cos^3 x}{6}+\frac{3}{6}\int\sin^2 x\cos^2 x\,dx \qquad (3)$$

Again putting p = 2 and q = 2 in (3), we have

$$\int\sin^2 x\cos^2 x\,dx=-\frac{\sin x\cos^3 x}{4}+\frac{1}{4}\int\sin^0 x\cos^2 x\,dx=-\frac{\sin x\cos^3 x}{4}+\frac{1}{4}\int\cos^2 x\,dx$$
$$=-\frac{\sin x\cos^3 x}{4}+\frac{1}{4}\int\left(\frac{1+\cos 2x}{2}\right)dx=-\frac{\sin x\cos^3 x}{4}+\frac{1}{8}\int(1+\cos 2x)\,dx$$
$$=-\frac{\sin x\cos^3 x}{4}+\frac{1}{8}\left(x+\frac{\sin 2x}{2}\right)$$

Thus (1) becomes: $\int\sin^4 x\cos^2 x\,dx=-\frac{\sin^3 x\cos^3 x}{6}+\frac{1}{2}\left[-\frac{\sin x\cos^3 x}{4}+\frac{1}{8}\left(x+\frac{\sin 2x}{2}\right)\right]$

$$=-\frac{\sin^3 x\cos^3 x}{6}-\frac{\sin x\cos^3 x}{8}+\frac{1}{16}\left(x+\frac{\sin 2x}{2}\right).$$

REMARK: We know that sin 0 = sin π = sin 2 π = cos π/2 = 0. Thus, if the limits of integration are taken from 0 to π/2 or 0 to π or 0 to 2 π, the first term on right side becomes zero. In this case, we have:

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{(p-1)}{(p+q)} \int_0^{\pi/2} \sin^{p-2} x \cos^q x\, dx$$

$$\int_0^{\pi} \sin^p x \cos^q x\, dx = \frac{(p-1)}{(p+q)} \int_0^{\pi} \sin^{p-2} x \cos^q x\, dx$$

$$\int_0^{2\pi} \sin^p x \cos^q x\, dx = \frac{(p-1)}{(p+q)} \int_0^{2\pi} \sin^{p-2} x \cos^q x\, dx$$

If the power of cos x is reduced, we get

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{(q-1)}{(p+q)} \int_0^{\pi/2} \sin^p x \cos^{q-2} x\, dx$$

$$\int_0^{\pi} \sin^p x \cos^q x\, dx = \frac{(q-1)}{(p+q)} \int_0^{\pi} \sin^p x \cos^{q-2} x\, dx$$

$$\int_0^{2\pi} \sin^p x \cos^q x\, dx = \frac{(q-1)}{(p+q)} \int_0^{2\pi} \sin^p x \cos^{q-2} x\, dx$$

Example 03: Evaluate (i) $\int_0^{2\pi} \sin^6 x \cos^5 x\, dx$ **(ii)** $\int_0^{2\pi} \sin^6 x \cos^4 x\, dx$ **(iii)** $\int_0^{2\pi} \sin^5 x \cos^5 x\, dx$

Solution: (i) Using reduction formula on sinx and cos x alternatively, we get

$$\int_0^{2\pi} \sin^6 x \cos^5 x\, dx = \frac{6-1}{6+5} \int_0^{2\pi} \sin^4 x \cos^5 x\, dx = \frac{5}{11} \cdot \frac{5-1}{5+4} \int_0^{2\pi} \sin^4 x \cos^3 x\, dx$$

$$= \frac{20}{99} \cdot \frac{4-1}{4+3} \int_0^{2\pi} \sin^2 x \cos^3 x\, dx = \frac{20}{231} \cdot \frac{3-1}{3+2} \int_0^{2\pi} \sin^2 x \cos x\, dx = \frac{40}{1155} \cdot \frac{2-1}{2+1} \int_0^{2\pi} \sin^0 x \cos x\, dx$$

$$= \frac{40}{3465} [\sin x]_0^{2\pi} = \frac{40}{3465} [\sin 2\pi - \sin 0] = 0 \qquad \{\text{NOTE}: \sin 2\pi = \sin 0 = 0\}$$

(ii) Using reduction formula on sinx and cos x alternatively, we get

$$\int_0^{2\pi} \sin^6 x \cos^4 x\, dx = \frac{6-1}{6+4} \int_0^{2\pi} \sin^4 x \cos^4 x\, dx = \frac{5}{10} \cdot \frac{4-1}{4+4} \int_0^{2\pi} \sin^4 x \cos^2 x\, dx$$

$$= \frac{1}{2} \cdot \frac{3}{8} \cdot \frac{4-1}{4+2} \int_0^{2\pi} \sin^2 x \cos^2 x\, dx = \frac{3}{32} \cdot \frac{2-1}{2+2} \int_0^{2\pi} \sin^0 x \cos^2 x\, dx = \frac{3}{128} \int_0^{2\pi} \cos^2 x\, dx$$

$$= \frac{3}{128} \cdot \frac{2-1}{2} \int_0^{2\pi} \cos^0 x\, dx = \frac{3}{256} \int_0^{2\pi} 1\, dx = \frac{3}{256} [x]_0^{2\pi} = \frac{3\pi}{128}$$

(iii) Using reduction formula on sinx and cos x alternatively, we get

$$\int_0^{2\pi} \sin^5 x \cos^5 x\, dx = \frac{5-1}{5+5} \int_0^{2\pi} \sin^3 x \cos^5 x\, dx = \frac{4}{10} \cdot \frac{5-1}{5+3} \int_0^{2\pi} \sin^3 x \cos^3 x\, dx$$

$$= \frac{2}{5} \cdot \frac{4}{8} \cdot \frac{3-1}{3+2} \int_0^{2\pi} \sin x \cos^3 x\, dx = \frac{1}{5} \cdot \frac{2}{5} \cdot \frac{3-1}{3+1} \int_0^{2\pi} \sin x \cos x\, dx$$

$$=\frac{2}{25}\cdot\frac{2}{4}\left[\frac{\sin^2 x}{2}\right]_0^{2\pi}=\frac{1}{50}\left[\sin^2 2\pi-\sin^2 0\right]=0 \qquad [\text{NOTE}: \sin 2\pi=\sin 0=0]$$

9.5 APPLICATIONS OF DEFINITE INTEGRATION

In this section we present a variety of applications of definite integration. These include "Volume of Solid of Revolution", "Area between Two Curves", "Average value of a function"

Application-I Volume of Solid of Revolution

Definite integrals are used to find the **volume of a solid of revolution**. The solid is produced by revolving a plane region or a curve about a line such as the x – axis. For illustration, consider the plane region shown in figure 1. Imagine that it is revolving about the x – axis. As it spins around the x – axis, it sweeps out a three – dimensional figure – a solid of revolution (figure 2).

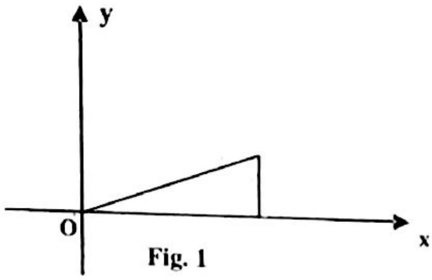


Fig. 1

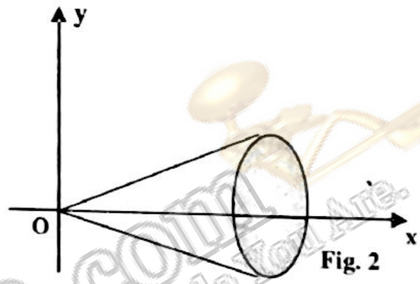


Fig. 2

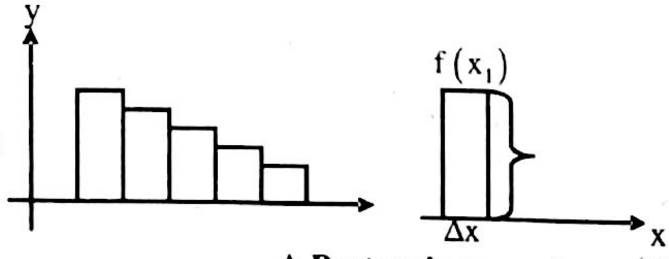
This particular solid is a cone. A formula for finding the volume of such solids of revolution is discussed next.

Consider a function y = f(x) that is non-negative and continuous on an interval [a, b]. Divide this interval into n equal subintervals of width Δx . We then have

$$\Delta x=\frac{b-a}{n}.$$

Draw the rectangles in each sub-interval. The heights of these rectangles then are $f(x_1), f(x_2)$, and so on (Figure 3). As the region is spun around the x – axis, each rectangle generates a cylinder. For example, the first rectangle with width Δx and height $f(x_1)$ generates a cylinder having radius $f(x_1)$ and height Δx (Figure 4).

Fig. 3



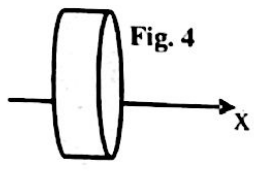


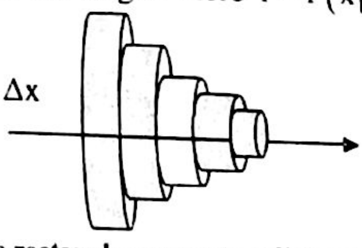
Fig. 4

A Rectangle generates a cylinder

The volume of a cylinder is $\pi r^2 h$, where r is the radius and h is the height. Here $r=f(x_1)$ and $h=\Delta x$. Thus the volume of the first cylinder generated is $\pi\{f(x_1)\}^2\Delta x$ · The volume of the second cylinder is $\pi\{f(x_2)\}^2\Delta x$

The total volume generated by revolving all rectangles is

$$\pi\{f(x_1)\}^2\Delta x+\pi\{f(x_2)\}^2\Delta x+\cdots+\pi\{f(x_n)\}^2\Delta x$$



The rectangles generate cylinders

In summation notation, we have

$$V = \sum_{i=1}^{n} \pi\{f(x_i)\}^2 \Delta x$$

This volume is approximately the volume of the solid of revolution. The larger n becomes, the smaller Δx becomes, and better is the approximation. The exact volume of the solid is the limit of this sum as $n \to \infty$ or as $\Delta x \to 0$.

$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{\infty} \pi\{f(x)\}^2 \Delta x$$

The limit on the right is the definite integral shown next.

The volume V of the solid produced by revolving the region bounded by y = f(x) and the x – axis (between x = a and x = b) about the x – axis is

$$V = \int_a^b \pi\{f(x)\}^2 dx$$

provided that f is continuous on [a, b].

Example 01: Find the volume of the solid of revolution obtained by revolving the curve $y = x^2$ about the x – axis between x = 1 and x = 3.

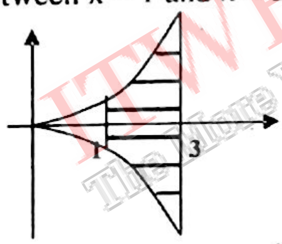
Solution: The volume of solid of revolution is given by:

$$V = \int_a^b \pi\{f(x)\}^2 dx$$

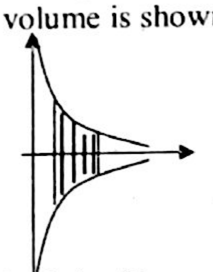
Substituting the values, we get

$$V = \int_1^3 \pi(x^2)^2 dx = \pi\int_1^3 x^4 dx = \pi\left(\frac{1}{5}\right)x^5\Big|_1^3 = \frac{1}{5}\pi(3^5 - 1^5) = \frac{1}{5}\pi(242) = 48.4\pi$$

Thus, the volume of the solid of revolution obtained by revolving the curve $y = x^2$ about the x – axis between x = 1 and x = 3 is 48.4π cubic units. This volume is shown in figure below.



Solid of Revolution of the curve $y = x^2$



Solid of Revolution of the curve $y = 1/x$

Example 02: Find the volume of the solid of revolution obtained by revolving the curve y = 1/x about the x – axis between x = 1 and x = 2.

Solution: Since $V = \int_a^b \pi\{f(x)\}^2 dx$. Substituting the values, we get

Hence the volume is $\pi/2$ cubic units. This volume is shown in the figure above.

Example 03: Find the volume of the solid of revolution obtained by revolving the curve $y = x^2 + 1$ about the x – axis on the interval [0, 3].

Solution: Here $V = \int_a^b \pi\{f(x)\}^2 dx$. Substituting the values, we get

$$V = \int_0^3 \pi(x^2+1)^2 dx = \pi\int_0^3 (x^4 + 2x^2 + 1) dx = \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^3$$

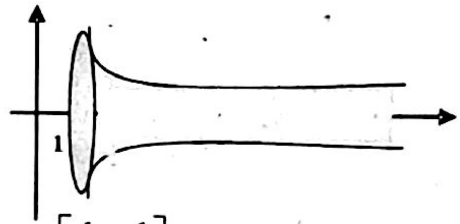
$$V=\pi\left[\frac{1}{5}(3)^5+\frac{2}{3}(3)^3+3-0\right]=\pi(48.6+18+3)=69.6\pi.$$

Hence the volume is $69.6\,\pi$ cubic units.

Example 04: Find the volume of Gabriel's Horn

Solution: It is the belief of Christians that at the day of Judgment that it is **Gabriel** who will below the horn "THE TRUMPET". But Muslims believe that it is **Israfiel** who will below this TRUMPET.

Any how, the function that is to be revolved about the x-axis is $f(x) = 1/x$ from $x = 1$ to $x = \infty$.
The figure is shown here.



Thus

$$V=\int_a^b \pi\{f(x)\}^2\,dx=\int_1^\infty \pi\left\{\left(\frac{1}{x}\right)\right\}^2=\pi\int_1^\infty x^{-2}\,dx=\pi\left[\frac{1}{-x}\right]_1^\infty=-\pi\left[\frac{1}{\infty}-\frac{1}{1}\right]=\pi$$

Application-II Area Between Two Curves

We know that one of the use of definite integration is to find the area under the curve $y = f(x)$ between the x-axis and the ordinates at say $x = a$ and $x = b$. The definite integration is also useful in finding the area between the two curves $y = f(x)$ and $y = g(x)$. This is shown graphically as under. In case if limits $x = a$ and $x = b$ are not given then both equations $y = f(x)$ and $y = g(x)$ are solved simultaneously to find two values of x. These will provide the limits under which required area is to be calculated.

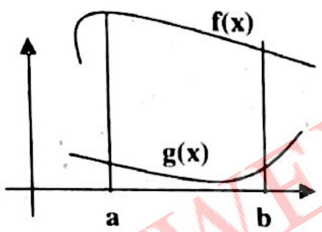


Fig. (1)

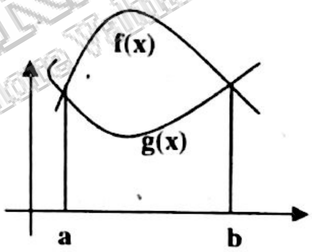


Fig. (2)

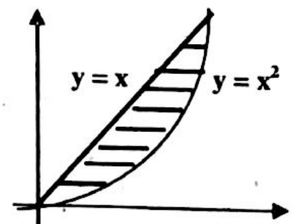
In figure 1, we se that the curves do not intersect whereas in the second figure the curves intersect each other at $x = a$ and $x = b$. In either case, the area between theses curves is computed by using the formula:

$$A=\int_a^b\{f(x)-g(x)\}\,dx$$

Example 05: Determine the area of the region enclosed by $y = x^2$ and $y = x$.

Solution: Since limits are not given hence we first solve two equations simultaneously to get the limits of integration.

Since $y = x^2$ and $y = x$ are tow curves, hence
$x^2 = x$ ➔ $x^2 - x = 0$ ➔ $x(x - 1) = 0$ ➔ $x = 0$ or $x = 1$. In this domain, if we plot both curves, we see that $y = x^2$ lies below the line $y = x$.
Here $y = x$ is the upper curve whereas $y = x^2$ is lower curve (This is shown in the figure). Thus,

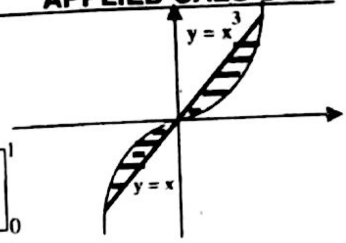


$$\text{Area}=\int_0^1 (x-x^2)\,dx=\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}\ \text{sq.units}$$

Example 06: Determine the area of the region enclosed by $y = x^3$ and $y = x$.

Solution: The required area in fact is the union of areas of two different regions. The first region starts from $x = -1$ to $x = 0$ where $y = x$ is the lower curve and $y = x^3$ is the upper curve.

The second region starts at $x = 0$ and ending at $x = 1$ where $y = x^3$ is the lower curve and $y = x$ is the upper curve. This is shown in the figure. Thus the required area is:



$$\text{Area} = \int_{-1}^{0}\left(x^3 - x\right)dx + \int_{0}^{1}\left(x - x^3\right)dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{1}$$

$$\text{Area} = 0 - 0 - \frac{1}{4}(-1)^4 + \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 - \frac{1}{4}(1)^4 - 0 + 0 = -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2} \text{ sq.units}$$

Example 07: Determine the area of the region enclosed by $y = x^3$ and $y = x^2$.

Solution: Solving two equations simultaneously, we get:
$x^3 = x^2$ ➔ $x^3 - x^2 = 0$ ➔ $x^2 (x - 1) = 0$ ➔ $x = 0$ or $x = 1$. In this interval, the graph of the curve $y = x^3$ lies below the curve $y = x^2$, so the required area is:

$$\text{Area} = \int_{0}^{1}\left(x^2 - x^3\right)dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{1} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \text{ sq.units}$$

Example 08: Determine the area of the region enclosed by $y = 5x - 6$ and $y = x^2$.

Solution: Solving the two equations simultaneously, we get:
$x^2 = 5x - 6$ ➔ $x^2 - 5x + 6 = 0$ ➔ that $x = 2$ and $x = 3$. Therefore the required area is given by:

$$\text{Area} = \int_{2}^{3}\left(5x - 6 - x^2\right)dx = \left[\frac{5}{2}x^2 - 6x - \frac{1}{3}x^3\right]_{2}^{3}$$

$$= \left[\frac{5}{2}(9) - 6(3) - \frac{1}{3}(27)\right] - \left[\frac{5}{2}(4) - 6(2) - \frac{1}{3}(8)\right] = \frac{45}{2} - 18 - 9 - 10 + 12 + \frac{8}{3} = \frac{1}{6} \text{ sq units}$$

Example 09: Determine the area between the curve $y = x^2 - 4$ and x – axis from $x = 1$ to $x = 2$.

Solution: The required area is given by:

$$\text{Area} = \int_{1}^{2}\left[0 - \left(x^2 - 4\right)\right]dx = \int_{1}^{2}\left(4 - x^2\right)dx = \left[4x - \frac{1}{3}x^3\right]_{1}^{2} = \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right) = \frac{5}{3} \text{ sq.units}$$

Application III Average Value of Function and Miscellaneous Applications

The average value of a function over the interval [a, b] is given by: $\frac{1}{b-a}\int_{a}^{b} f(x)\,dx$

Example 10: Blood does not flow with a constant velocity. It flows fastest in the centre of an artery and slowest next to the wall of the artery. In fact, the velocity v at any distance x from the centre can be expressed as a function of x. Let us assume that for an artery of radius 0.2 cm, the velocity is $v = 40 - 990\,x^2$. Find the average velocity of blood.

Solution: The average velocity is given by

$$\bar{v} = \frac{1}{0.2 - 0}\int_{0}^{0.2}(40 - 990x^2)\,dx = 5\left[40x - 990\frac{x^3}{3}\right]_{0}^{0.2} = 26.8 \text{ cm/s}$$

Example 11: A ball is dropped from a high altitude balloon. If the ball falls with velocity $v = 32\,t$ ft/s, how far the ball travel during the first 4 seconds?

Solution: The total distance traveled by ball is given by

$$s = \int_{0}^{4} v\,dt = 32\int_{0}^{4} t\,dt = 32\left[\frac{t^2}{2}\right]_{0}^{4} = 16\left(4^2 - 0^2\right) = 256 \text{ ft}$$

WORKSHEET 09

1. The rate at which petroleum was consumed in India was approximately $c'(t) = 21t + 281$ million barrels per year from 1983 (t = 0) to 1987 (t = 4). Determine the total amount of petroleum consumed from 1983 to 1987.

2. A man starts his car and then drives it with a constant acceleration of 18 feet per second per second. How far does the car go in 5 seconds?

3. Find the volume of the solid produced by revolving about the x – axis the region whose boundary is given. (Note: $y = 0$ is the equation of the x – axis.)

(a) $y = x^2 + 1, y = 0, x = 0, x = 1$. (b) $y = \frac{1}{\sqrt{x}}$, the x - axis, $x = e, x = 10$.

(c) $y = x^2 - 5x, y = 0, x = 5.1, x = 5.5$.

4. Determine the area enclosed between the two curves. Graphing is recommended:

(i) $y = x + 5,\ y = \sqrt{x}$ from $x = 0$ to $x = 4$ (ii) $y = x^3,\ y = x^2$

(iii) $y = 1 - x^2,\ y = x + 4$ from $x = -3$ to $x = 1$ (iv) $y = e^x,\ y = 1$ and $x = 1$

5. Determine the area enclosed by the curves. In each case, the region enclosed consists of two regions. A careful study of the graphs is essential.

(i) $y = x^3,\ y = -x$ (ii) $y = x^4,\ y = x^2$

6. Let t be the number of years from now. What is the total amount of waste dumped into a lake by a factory that dumps waste at the rate of $5e^{-0.1t}$ pounds per year indefinitely?

7. Evaluate: (i) $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ (ii) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

8. Show that:

(i) $\int_0^{\pi/2} \ln(\tan x) dx = 0$ (ii) $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$

(iii) $\int_0^1 \ln\left(\frac{1}{x} - 1\right) dx = 0$ (iv) $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2$.

9. Discuss the convergence of

(i) $\int_1^{\infty} \frac{x}{(1+x)^3} dx$ (ii) $\int_0^{\infty} x e^{-x^2} dx$ (iii) $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

(iv) $\int_1^2 \frac{x dx}{\sqrt{x-1}}$ (v) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} dx$.

10. Evaluate:

(i) $\int_0^{\infty} x^3 e^{-x^3} dx$ (ii) $\int_0^{\infty} \sqrt{x} e^{-x^3} dx$ (iii) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x dx$

(iv) $\int_0^{\pi/2} \sin^7 x dx$ (v) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

11. Use the definition of Gamma Function, evaluate the following

(i) $\int_0^{\infty} \sqrt{x} e^{-x} dx$ (ii) $\int_0^{\infty} x^4 e^{-x^2} dx$ (iii) $\int_0^{\infty} e^{-h^2 x^2} dx$

CHAPTER TEN

VECTOR ANALYSIS

10.1 INTRODUCTION

In this introductory section, we shall be concerned with defining some basic and important concepts of scalars and vectors. Some definitions and concepts will also be given which play considerable important role in the vector analysis.

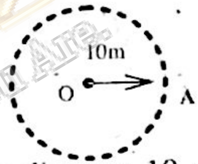
Definition: A **scalar** is a quantity having magnitude but no direction along with some unit of measurement.

For example; mass, length, time, temperature, density, distance, area, and volume.

Scalars are indicated by letters in ordinary type as in elementary algebra. Operations with scalars follow the same rules as in elementary algebra.

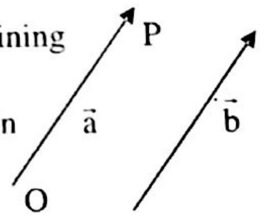
Definition: A **vector** is a quantity having magnitude and direction as well as some unit of measurement.

For example, displacement, force, velocity and acceleration are vectors.



Suppose, we are told that a person moves a distance of 10 m from a point O on a circular path then he may be any where on the circumference of a circle exactly 10 m apart from the centre (see fig). This means that the distance 10 m is not sufficient to locate the position of the person. Therefore, some additional information about direction of motion must be supplied along with the distance. Now if we say that the person has moved 10 m East of O, then his position is precisely located and the displacement vector $\overline{OA}$ is completely determined.

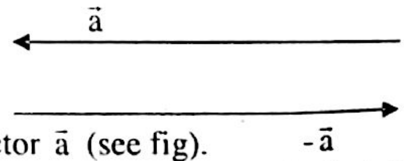
Graphical Representation of a Vector



Graphically, a vector is represented by an arrow $\overrightarrow{OP}$ (see fig) defining the direction and magnitude of the vector being indicated by the length of the arrow. The tail end O of the arrow is called the origin or initial point of the vector, and the head P is called the terminal point.

Analytically, a vector is represented by a letter with an arrow over it, such as $\vec{a}$ (see fig) and its magnitude or length is denoted by $|\vec{a}|$.

Definition: Two vectors $\vec{a}$ and $\vec{b}$ are said to be **equal** if they have the same magnitude and direction regardless of the position of their initial points. In this case we write $\vec{a} = \vec{b}$ (see the above fig).



Definition: A vector having the direction opposite to that of vector $\vec{a}$ but having the same magnitude is denoted by $-\vec{a}$ and is called the '**negative**' of the vector $\vec{a}$ (see fig).

Definition: The vector of zero magnitude is called the **null or zero vector**. In fact, the null vector is an imaginary vector and it may have any direction.

Definition: The vector whose magnitude is one is called the **unit** vector. If we divide the given vector $\vec{a}$ by its magnitude then we obtain the unit vector $\hat{a}$ in the direction of vector $\vec{a}$. For example, the unit vector along the vector $\vec{a}$ is given by $\hat{a} = \vec{a}/|\vec{a}|$.

This definition of unit vector suggests that any vector $\vec{a}$ can be represented by the product of a unit vector $\hat{a}$ in the direction of $\vec{a}$ and the magnitude of $\vec{a}$ Symbolically, this is written as:

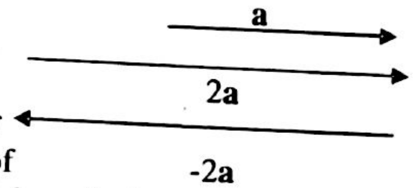
$$\vec{a} = |\vec{a}|\hat{a}$$

REMARK: In what follows, we shall use bold face small case letters to indicate the vector e.g; **a**, **b**, **c**, etc are the vectors and a, b, c denote their respective magnitudes. The unit vector will be denoted by **u**.

Definition: Two nonzero vectors **a** and **b** are said to be **collinear** if there exists a nonzero real number k such that **a** = k **b**.

If $k > 0$ then the vectors **a** and **b** have the same direction.

If $k < 0$ then the vectors **a** and **b** have opposite directions.



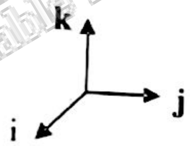
If a vector **a** is multiplied by a scalar k, then k**a** is a vector parallel to **a** with length |k| times that of **a**. The direction of k**a** remains unchanged if $k > 0$ and it is reversed if $k < 0$. In the above figure, the multiplication of the vector **a** by 2 and -2 is shown.

Three Dimensional Coordinates

Throughout these sections, we shall be concerned with vectors in three-dimensional space. By the introduction of three mutually perpendicular axes, with the same unit of length along all three axes, we obtain the usual Cartesian coordinate system. The conventional orientation of axes is shown in the following figure.

Rectangular Unit Vectors and Their Representation

The vectors **i** = [1, 0, 0], **j** =[0, 1, 0] and **k** = [0, 0, 1] are known as rectangular unit vectors. These are depicted in the figure shown.



Now any non-zero vector **a** = [a_1, a_2, a_3] in R^3 can be expressed as:

a = [a_1, a_2, a_3] = [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3]
= a_1 [1, 0, 0] + a_2 [0, 1, 0] + a_3 [0, 0, 1] = a_1 **i** + a_2 **j** + a_3 **k**

Example 01: Show that the vectors a = (4/3) i – 2j + 3k and b = -2i + 3j – (9/2) k are collinear vectors.

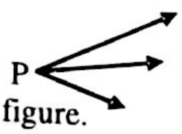
Solution: If vector **a** and **b** are collinear vectors we must have: **a** = m **b**.

➔ (4/3) **i** – **j** + 3**k** = m(-2**i** + 3**j** – (9/2) **k**) ➔ 4/3 = -2m, -2 = 3m and 3 = (-9/2) m

From three equations we observe that m = -2/3. Thus **a** = (-2/3) **b**. Hence two vectors are collinear.

Definition: The vectors lying in the same plane are called the **coplanar vectors** otherwise they are **non–coplanar vectors.**

Definition: The vectors intersecting at a single point are called the **concurrent vectors** and point where the vectors intersect is called the **point of concurrency**. Three concurrent vectors are shown in the adjacent figure.



If $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ are points in space, the vector represented by $\overrightarrow{P_1P_2}$ is

$\overrightarrow{P_1P_2} = (x_2 - x_1)$ **i** + $(y_2 - y_1)$ **j** + $(z_2 - z_1)$ **k**

Definition: The **magnitude or length** of the vector **a** = [a_1, a_2, a_3] = a_1 **i** + a_2 **j** + a_3 **k** is defined to be a number $|a| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Definition: The **unit vector** along a space vector **a** = x**i** + y**j** + z **k** is given by

$$\mathbf{u} = \frac{\vec{a}}{|a|} = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$$

Example 02: Find the magnitude of the vector a = 5i – 3j + 9k. Also find the unit vector along a.

Solution: The magnitude of the given vector **a** is

$$|a| = |5\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}| = \sqrt{5^2 + (-3)^2 + 9^2} = \sqrt{25+9+81} = \sqrt{115}$$

Also, the unit vector along **a** is given by : $\mathbf{u} = \frac{\vec{a}}{|a|} = \frac{5}{\sqrt{115}}\mathbf{i} - \frac{3}{\sqrt{115}}\mathbf{j} + \frac{9}{\sqrt{115}}\mathbf{k}$

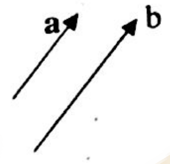
Example 03: Find the vector whose magnitude is 5 and is in the direction of the vector 4i – 3j + k.

Solution: Let **a** = 4**i** – 3**j** + **k**, |**a**| = $\sqrt{4^2+(-3)^2+1^2} = \sqrt{26}$. Thus, unit vector in the direction of **a** is: $\mathbf{u} = \frac{\vec{a}}{|a|} = \frac{4}{\sqrt{26}}\mathbf{i} - \frac{3}{\sqrt{26}}\mathbf{j} + \frac{1}{\sqrt{26}}\mathbf{k}$

If **b** is a vector of magnitude 5 and is in the direction of vector a then,

$$\mathbf{b} = 5\mathbf{a} = \frac{12}{\sqrt{26}}\mathbf{i} - \frac{15}{\sqrt{26}}\mathbf{j} + \frac{5}{\sqrt{26}}\mathbf{k}$$

Definition: Two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{a} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are said to be parallel if the ratio of their components is same.
Thus the condition that two vector as shown above are parallel if



$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

This is shown in the above figure.

Example 04: Find the value of λ if the vectors 6i + j - k and λ I – 4j + 4k are parallel.

Solution: Since the given vectors are parallel, so $\frac{\lambda}{6} = \frac{-4}{1} = \frac{4}{-1} \Rightarrow \lambda = (6)(-4) = -24$.

10.2 PRODUCT OF TWO VECTORS

When two scalars are multiplied, the result is a scalar, but when two vectors are multiplied, the result may be scalar or vector. Thus, the product of two vectors is of two kinds. These are **(i)** Scalar Product and **(ii)** Vector Product.

Scalar or Dot Product of Two Vectors

The product of two vectors that produces a scalar quantity is known as scalar product of two vectors. Since this product is shown by placing a dot (.) between the two vectors hence, it is also known as a Dot Product. **For example, if a force F is applied on a body which is displaced through the vector d then the product F . d** is called work done by the force **F**. This is denoted by:

$$W = \mathbf{F} \cdot \mathbf{d}$$

Definition 1: If **a** and **b** are two non-zero vectors at an angle θ, their scalar product is defined as:

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta \text{ where } 0 \le \theta \le \pi$$

Here a and b are the magnitudes of vectors **a** and **b** respectively.

Definition 2: If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then dot product between **a** and **b** is also defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

From the two definitions, we conclude that

$$ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

This implies that: $\theta = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{ab}\right) = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$

Example 01: A particle is acted on by constant forces 4i + j -3k and 3i + j - k which, is displaced from the point i + 2j + 3k to the point 5i + 4j + k. Find the total work done by the forces.

Solution: Let $\mathbf{F_1} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{F_2} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ then the total force is

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} = 7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

If **d** is the displacement measured from the point **i** + 2**j** + 3**k** to the point 5**i** + 4**j** + **k** then

$$\mathbf{d} = (5\mathbf{i} + 4\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

Thus, the required work done by the given forces is

$$W = \mathbf{F} \cdot \mathbf{d} = (7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 28 + 4 + 8 = 40 \text{ J.}$$

Example 02: Find the angle between the vectors a = I + 2j - k and b = -I + j – 2k.

Solution: Let θ be the angle between the vectors **a** and **b**. Then using the formula,

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

we have

$$\theta = \cos^{-1}\left(\frac{1(-1) + 2(1) + (-1)(-2)}{\sqrt{1^2 + 2^2 + (-1)^2}\sqrt{(-1)^2 + 1^2 + (-2)^2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

Useful Results

i . **i** = (1) (1) cos 0 = 1. Similarly, **j** . **j** = **k** . **k** = 1.

Also **i** . **j** = 1 . 1. cos 90° = 0. Similarly, **j** . **k** = **k** . **i** = 0.

Theorem: Two non-zero vectors are perpendicular if and only if their dot product is zero.

Proof: Let the vectors **a** and **b** be perpendicular, then

$$\mathbf{a} \cdot \mathbf{b} = ab \cos 90^\circ = 0 \qquad \text{(because } \cos 90^\circ = 0)$$

Now, let **a** . **b** = 0 ➔ ab cos θ = 0 ➔ cos θ = 0 ➔ θ = 90°.

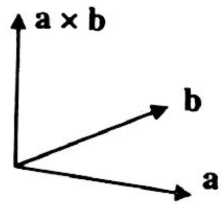
This implies that the vectors **a** and **b** are perpendicular.

Example 03: For what values of p the vectors a = [2, -1, 2] and b = [3, 2p, 0] are perpendicular?

Solution: Since the given vectors are perpendicular, therefore: **a** . **b** = 0. Now,

$$\mathbf{a} \cdot \mathbf{b} = [2, -1, 2] \cdot [3, 2p, 0] = 6 - 2p + 0 = 0 \rightarrow p = 3.$$

Vector or Cross Product of Two Vectors



The product of two vectors that produces a vector quantity is known as vector product of two vectors. Since this product is shown by placing a cross (×) between the two vectors, hence it is also known as a Cross Product. For example, let **F** be the applied force and **r** be the arm.

We define the Momentum or Torque which is a vector as: $\tau = \mathbf{F} \times \mathbf{r}$

Definition 1: If **a** and **b** are two non-zero vectors acting at an angle θ, their cross product is a vector quantity whose direction is perpendicular to both **a** and **b** and its magnitude is given by:

$$|\mathbf{a} \times \mathbf{b}| = a.b \sin\theta$$

Thus

$$\mathbf{a} \times \mathbf{b} = a.b \sin\theta\ \mathbf{n}$$

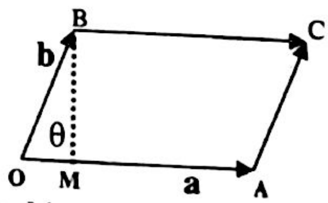
Here **n** is a normal vector to **a** and **b**.

Definition 2: The vector product between two non-zero vectors **a** and **b** is also defined as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem: Prove that the magnitude of the cross product of two non - zero vectors **a** and **b** represents thearea of parallelogram having **a** and **b** as adjacent sides, that is,

Area of parallelogram = |**a** × **b**|.



Proof: We know that

Area of parallelogram = Length × Altitude = |OA| |BM|.

From the figure, we see that: sin θ = |BM| / |OB|

➔ |BM| = |OB| sin θ

Therefore, area of parallelogram = |OA| |OB| sin θ = a b sin θ = |**a** × **b**|

Example 04: If a = 5i – 3j + 4k and b = 19i + 7j - k, then find a × b.

Solution: By definition,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 5 & -3 & 4 \\ 19 & 7 & -1 \end{vmatrix} = \mathbf{i}\,(3-28) - \mathbf{j}\,(-5-76) + \mathbf{k}\,(35+57) = -25\mathbf{i} + 81\mathbf{j} + 92\mathbf{k}$$

Example 05: Find the area of a parallelogram whose adjacent sides are i – 2j + 3k and 2i + j – 4k.

Solution: Let **a** = **i** – 2**j** + 3**k** and **b** = 2**i** + **j** – 4**k**. We know that area of parallelogram is given by | **a** × **b**|. Now,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix} = \mathbf{i}\,(8-3) - \mathbf{j}\,(-4-6) + \mathbf{k}\,(1+4) = 5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$$

$$\Rightarrow |\mathbf{a}\times\mathbf{b}| = \sqrt{(5)^2+(10)^2+(5)^2} = \sqrt{150} = 5\sqrt{6}.$$

Thus, area of the parallelogram whose adjacent sides are vectors **a** and **b** is $5\sqrt{6}$ unit^2.

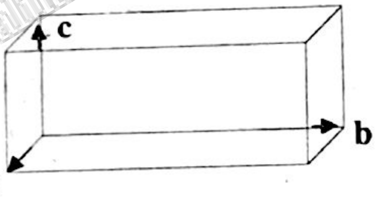
REMARK: The area of a triangle = 1/2 (The area of parallelogram) = 1/2 |**a** × **b** |.

Scalar Triple Product or Box Product

Let **a, b** and **c** be three vectors. Then their scalar triple product is denoted by a . (**b** × **c**) and is defined as:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]$$

Geometrical Interpretation of Scalar Triple Product



A solid figure with six faces that are parallelograms is called **parallelepiped**. In a rectangular parallelepiped, the faces are rectangles. If the faces are squares, the parallelepiped is a **cube**.

The volume of parallelepiped is given by "Scalar Triple Product".

That is, $V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

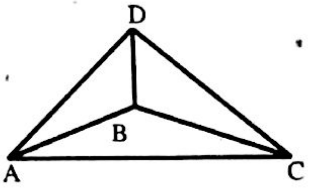
EXAMPLE 06: Find the volume of parallelepiped if a = -3i + 7j + 5k, b = -3i + 7j – 3k and c = 7i -5j – 3k are its edges.

Solution: Volume of a parallelepiped

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3(-21-15) - 7(9+21) + 5(15-49) = -272$$

Since the volume is always positive, therefore volume of required parallelepiped is 272 unit^3.

Tetrahedron (Triangular Pyramid)



A solid figure bounded by four triangular faces is called *tetrahedron*.

Volume of Tetrahedron

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are the vertices of a tetrahedron then its volume can be found in the following way:

$$V_T = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

It may also be noted that if six tetrahedrons of same sizes are combined together they form a parallelepiped. Thus, $V_T = 1/6$ (Volume of Parallelepiped)

Now in the above figure, if we let **a** = AB, **b** = AC and **c** = AD then

$$V_T = 1/6\,[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]$$

Example 07: Show that the volume of the tetrahedron whose vertices are (0, 1, 2), (3, 0, 1), (4, 3, 6) and (2, 3, 2) is 6.

Solution: Method-I. Using the formula, $V_T = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

We obtain, $V_T = \frac{1}{6}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 3 & 6 & 1 \\ 2 & 3 & 2 & 1 \end{vmatrix} = \frac{1}{6}\left[0\begin{vmatrix} 0 & 1 & 1 \\ 3 & 6 & 1 \\ 3 & 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 & 1 \\ 4 & 6 & 1 \\ 2 & 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 0 & 1 \\ 4 & 3 & 6 \\ 2 & 3 & 2 \end{vmatrix}\right]$

$$= \frac{1}{6}\left[-\{3(6-2)-(4-2)+(8-12)\}+2\{3(3-3)+(12-6)\}-\{3(6-18)+(12-6)\}\right] = \frac{1}{6}(36) = 6\text{unit}^3$$

Method-II: **a** = **AB** = [3 – 0, 0 – 1, 1 – 2] = [3, -1, -1],
b = **AC** = [4 – 0, 3 – 1, 6 – 2] = [4, 2, 4],
c = **AD** = [2 – 0, 3 – 1, 2 – 2] = [2, 2, 0].

Now $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{vmatrix} = 3(0-8)+1(0-8)-1(8-4) = -24-8-4 = -36$

Thus volume of Tetrahedron is $V_T = (1/6)\,[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] = (1/6)(36) = 6$ unit^3.

This result is same as above. Here negative sign is neglected because volume is always positive.

REMARK: We know that if two rows of a determinant are interchanged the value of determinant is changed by negative sign. If this interchange is done twice the value of determinant remains same.

Now, $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Thus, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ OR $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

Vector Triple Product

If **a**, **b** and **c** are three nonzero vectors, then their vector triple product is usually denoted by $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ or $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and are defined as follows:

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ **AND** $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\,\mathbf{a}$

It is clear from the above definitions that: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Example 08: If a = i + j + 3k, b = -i + 7j -2k and c = 2i – j + k then verify the formula,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

Solution: Consider,

$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} i & j & k \\ -1 & 7 & -2 \\ 2 & -1 & 1 \end{vmatrix} = i(7-2) - j(-1+4) + k(1-14) = 5i - 3j - 13k$$

Thus, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 5 & -3 & -13 \end{vmatrix} = i(-13+9) - j(-13-15) + k(-3-5) = -4i + 28j - 8k$ (1)

Taking right hand side, we get

$(\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} = \{(1)(2)+(1)(-1)+(3)(1)\}(-i+7j-2k) - \{(1)(-1)+(1)(7)+(3)(-2)\}(2i-j+k)$

$= (4)(-i+7j-2k) - (0)(2i-j+k) = -4i + 28j - 8k$ (2)

From (1) and (2), we see that : $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$

Scalar Product of Four Vectors

If **a**, **b**, **c** and **d** are any four nonzero vectors, then the scalar product of **a** × **b** with **c** × **d** denoted by (**a** × **b**) . (**c** × **d**) is known as scalar product of four vectors. To evaluate this product, we first compute (**a** × **b**) and then (**c** × **d**). Finally, we compute their dot product to get the required scalar product of four vectors.

Vector Product of Four Vectors

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10.3 VECTOR FUNCTIONS

If to each value of a scalar variable t in some range there corresponds a unique vector **f** in space, then **f** is said to be a vector function of t. This is denoted by **f**(t). If a Cartesian system of coordinates is chosen, then we can write: $\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$

where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are three scalar functions and are the components of **f**(t) along x – axis, y – axis and z – axis, respectively. $\mathbf{f}(t) = \cos t\,\mathbf{i} + t^2\,\mathbf{j} + \sin t\,\mathbf{k}$ and $\mathbf{g}(t) = 5t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k}$ are examples of vector functions.

Differentiation of Vector Function

Let **f**(t) be a continuous vector function of a scalar variable t. Then if the limit

$$\lim_{\Delta t \to 0} [\mathbf{f}(t + \Delta t) - \mathbf{f}(t)] / \Delta t$$

exists, it is called the derivative of **f** with respect to t and is denoted by **df/dt** or $\mathbf{f}'(t)$. If **df/dt** exists at $t = t_0$, **f** is said to be differentiable at $t = t_0$.

The function **df/dt** is itself a vector function of t and its derivative, if it exists, is called the second derivative of **f**(t) and is denoted by $d^2\mathbf{f}/dt^2$ or $\mathbf{f}''(t)$ and is defined as:

$$d^2\mathbf{f}/dt^2 = \lim_{\Delta t \to 0} [\mathbf{f}'(t+\Delta t) - \mathbf{f}'(t)] / \Delta t$$

Similarly, we define higher derivatives namely, $\mathbf{f}'''(t)$, $\mathbf{f}^{iv}(t)$, …, $f^{n}(t)$.

$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$ is derivable then **f** is differentiable if and only if the scalar functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are differentiable and in this case,

$$\mathbf{f}'(t) = f_1'(t)\,\mathbf{i} + f_2'(t)\,\mathbf{j} + f_3'(t)\,\mathbf{k}$$

Vector Rules of Differentiation

Let **f** and **g** are vector functions of t, then

(i) $\frac{d}{dt}[c\,\mathbf{f}(t)] = c\,\mathbf{f}'(t)$, c being scalar

(ii) $\frac{d}{dt}(\mathbf{f} \pm \mathbf{g}) = \mathbf{f}'(t) \pm \mathbf{g}'(t)$

(iii) $\frac{d}{dt}(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}(t) \cdot \mathbf{g}'(t) + \mathbf{f}'(t) \cdot \mathbf{g}(t)$

(iv) $\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \mathbf{f}(t) \times \mathbf{g}'(t) + \mathbf{f}'(t) \times \mathbf{g}(t)$

REMARK: In (iv) the order of the functions **f**, **g** and their derivatives is not to be changed because cross product is not commutative.

(ii) The position vector is usually denoted by **r** and is defined as: **r** = x**i** + y**j** + z**k** where x, y and z are functions of variable t.

Example 01: A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the magnitude of velocity and acceleration of the particle at time t = 1 where t in seconds and x, y and z are in meters.

Solution: The position vector **r** = x**i** + y**j** + z**k** = $(t^3 + 1)\,\mathbf{i} + t^2\mathbf{j} + (2t + 5)\,\mathbf{k}$ 5. Differentiating with respect to *t*, we get

Velocity $= \mathbf{v} = d\mathbf{r}/dt = 3t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2\mathbf{k}$

At t = 1, we have $\mathbf{v} = 3\,\mathbf{i} + 2\,\mathbf{j} + 2\mathbf{k}$

Acceleration $= \mathbf{a} = d\mathbf{v}/dt = 6\,t\,\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$

At t = 1, $\mathbf{a} = 6\mathbf{i} + 3\mathbf{j}$

Thus, $|\mathbf{v}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$ m/s and $|\mathbf{a}| = \sqrt{6^2 + 2^2 + 0^2} = \sqrt{40}$ m/s^2

Integration of Vector Functions

Let $\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$. Suppose $f_1(t)$, $f_2(t)$, and $f_3(t)$ are continuous in a specified interval. Then the equation $\int \mathbf{f}(t)dt = \int f_1(t)dt\,\mathbf{i} + \int f_2(t)dt\,\mathbf{j} + \int f_3(t)dt\,\mathbf{k}$ defines an indefinite integral of **f**(t). If there exists a vector function **g**(t) such that

$$\mathbf{f}(t) = \frac{d}{dt}\mathbf{F}(t) \text{ then } \int \mathbf{f}(t)dt = \mathbf{F}(t) + \mathbf{c}$$

where **c** is an arbitrary constant vector independent of t.

The definite integral of a vector function **f**(t) between the limits t = a and t = b can be written as:

$$\int_a^b \mathbf{f}(t)dt = \mathbf{F}(b) - \mathbf{F}(a)$$

Example 02: Determine the vector function which has 2cos 2t i + 2sin 2t j + 4k as its derivative and 2i – 3j + k as its value at t = 0.

Solution: Let $\mathbf{f}'(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 4\mathbf{k}$. Integrating,

$$\int \mathbf{f}'(t)dt = \int (2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 4t\,\mathbf{k}) + \mathbf{c}$$

➔ $\mathbf{f}(t) = \sin 2t\,\mathbf{i} - \cos 2t\,\mathbf{j} + 4t\,\mathbf{k} + \mathbf{c}$ (1)

Putting t = 0 and $\mathbf{f}(t) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, we get: $2\mathbf{i} - 3\mathbf{j} + \mathbf{k} = 0(\mathbf{i}) - (1)\mathbf{j} + (0)\mathbf{k} + \mathbf{c}$

➔ $\mathbf{c} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Substituting in (1), we get: $\mathbf{f}(t) = \sin 2t\,\mathbf{i} - \cos 2t\,\mathbf{j} + 4t\,\mathbf{k} + 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

➔ $\mathbf{f}(t) = (2 + \sin 2t)\,\mathbf{i} - (2 + \cos 2t)\,\mathbf{j} + (1 + 4t)\,\mathbf{k}$

This is the required vector function.

Example 03: A particle moves such that its acceleration is given by: $\mathbf{f}''(t) = (t^4 + 2t)\,\mathbf{i} + t^2\,\mathbf{j} - t\,\mathbf{k}$. Find the velocity and displacement vector given that $\mathbf{f}'(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{f}(t) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ at t = 0.

Solution: We know that acceleration is a rate of change of velocity, that is;

$$\mathbf{a}\,\frac{d}{dt}(\mathbf{v}) = \mathbf{f}''(t) = (t^4 + 2t)\,\mathbf{i} + t^2\,\mathbf{j} - t\,\mathbf{k}$$

Integrating, we get $\int \mathbf{f}''(t)dt = \int [(t^4 + 2t)\,\mathbf{i} + t^2\,\mathbf{j} - t\mathbf{k}]\,dt + \mathbf{c}$

➔ $\mathbf{v} = \mathbf{f}'(t) = \left(\frac{t^5}{5} + t^2\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{c}$ (1)

To find **c**, put t = 0 in (1), we obtain: $\mathbf{f}'(0) = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} + \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

➔ $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Thus equation (1) becomes

$$\mathbf{v} = \mathbf{f}'(t) = \left(\frac{t^5}{5} + t^2 + 1\right)\mathbf{i} + \left(\frac{t^3}{3} + 1\right)\mathbf{j} - \left(\frac{t^2}{2} + 1\right)\mathbf{k} \qquad (2)$$

This is the required velocity vector. Now, we know that

$\mathbf{v} = \mathbf{f}`(t) = d\mathbf{r}/dt = \left(\frac{t^5}{5} + t^2 + 1\right)\mathbf{i} + \left(\frac{t^3}{3} + 1\right)\mathbf{j} - \left(\frac{t^2}{2} + 1\right)\mathbf{k}$, where $\mathbf{r}$ is a position vector.

Integrating, we get: $\mathbf{r} = \int\left(\frac{t^5}{5} + t^2 + 1\right)dt\ \mathbf{i} + \int\left(\frac{t^3}{3} + 1\right)dt\ \mathbf{j} - \int\left(\frac{t^2}{2} + 1\right)dt\ \mathbf{k} + \mathbf{d}$

$$= \left(\frac{t^6}{30} + \frac{t^3}{3} + t\right)\mathbf{i} + \left(\frac{t^4}{12} + t\right)\mathbf{j} - \left(\frac{t^3}{6} + t\right)\mathbf{k} + \mathbf{d} \qquad (3)$$

Here **d** is a constant of integration.

To find **d** put t = 0 in (3), we get: $\mathbf{r} = \mathbf{f}(0) = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} + \mathbf{d} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

➔ $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Thus equation (3) becomes

$$\mathbf{r} = \left(\frac{t^6}{30} + \frac{t^3}{3} + t + 1\right)\mathbf{i} + \left(\frac{t^4}{12} + t + 2\right)\mathbf{j} - \left(\frac{t^3}{6} + t + 1\right)\mathbf{k}$$

This is the required displacement vector.

10. 4 DEL – THE DIFFERENTIAL OPERATOR

Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of a point P(x, y, z) in a given region of the space $\mathbf{R}^3$. A function **f** which, associates a unique vector **f(r)** with each vector **r** in the given region of space is called **vector point function**. We can write **f(r)** as **f**(x, y, z).

A function ϕ which associates a unique scalar $\phi(\mathbf{r})$ with each vector **b** in a given region of space is called **scalar point function**. We may write $\phi(\mathbf{r})$ as $\phi(x, y, x)$.

Domain of a vector point function is called **vector field** and domain of a scalar point function is called a **scalar field**.

Definition: The vector differential operator ∇, is called **del** or **nebula**, and is defined as

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

This differential operator has an important role in three dimensional physical problems.

The Gradient

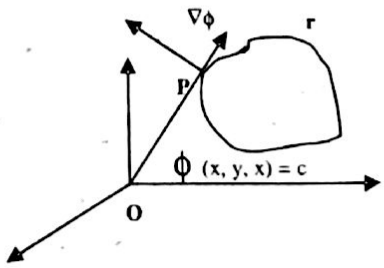
Let □ (x, y, x) be a scalar point function with domain D. Suppose $\partial\phi/\partial x$, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ are continuous in D. Then the gradient of □ is written as $\nabla\phi$ or grad ϕ and is defined as

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(\phi) \Rightarrow \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

The Directional Derivative

The directional derivative of a scalar function ϕ (x, y, x) in the direction of vector a is defined as:

$$\frac{\partial\phi}{\partial s} = (\nabla\phi).\mathbf{a}/|\mathbf{a}|$$



Geometrical Interpretation of Gradient

Consider a surface $\phi(x, y, x) = c$. Let P(x, y, z) be any point on this surface. Join OP and let **OP = r**.

Since, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ ➔ $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

Also, the differential of $\phi(x, y, x) = c$ is given by:

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0 \Rightarrow \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)(dx\,i + dy\,j + dz\,k) = 0$$

$$\Rightarrow (\nabla\phi).(dr) = 0$$

Now $\nabla\phi$ is the gradient of a scalar field function ϕ (x, y, x) which is tangential to the surface ϕ(x, y, x) = c and the dot product of $\nabla\phi$ and dr is zero, this implies that $\nabla\phi$ and **dr** are perpendicular to each other. Now, dr lies on the surface ϕ (x, y, x) = c, this shows that grad □ is perpendicular to the surface ϕ (x, y, x) = c.

Example 01: Find the gradient vector and its modulus of a scalar function $\phi(x,y,z) = x^3y^2 - z^2yx^2 - 9$ at (1, 2, 1).

Solution: We know that: $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3y^2 - z^2yx^2 - 9)$

➔ $$\nabla\phi = i\frac{\partial}{\partial x}(x^3y^2 - z^2yx^2 - 9) + j\frac{\partial}{\partial y}(x^3y^2 - z^2yx^2 - 9) + k\frac{\partial}{\partial z}(x^3y^2 - z^2yx^2 - 9)$$

$$= (3x^2y^2 - 2xyz^2)i + (2x^3y - x^2z^2)j + (-2x^2yz)k$$

At (1, 2, 1), $\nabla\phi = 8i + 3j - 4k$

Also , $|\nabla\phi| = \sqrt{8^2 + 3^2 + (-4)^2} = \sqrt{89}$

Example 02: Find the directional derivative of $4x^2y^2z^2$ at (1, 2, 1) in the direction of vector 2i + j + 2k.

Solution: Let $\phi(x,y,z) = 4x^2y^2z^2$, P = (1,2,1) and **a** = 2i + j + 2k. Now, the directional derivative is given by: $\frac{\partial\phi}{\partial s} = (\nabla\phi).a/|a|$ (1)

Now,

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4x^2y^2z^2) = i\frac{\partial}{\partial x}(4x^2y^2z^2) + j\frac{\partial}{\partial y}(4x^2y^2z^2) + k\frac{\partial}{\partial z}(4x^2y^2z^2)$$

➔ $$\nabla\phi = 8xy^2z^2\,i + 8x^2yz^2\,j + 8x^2y^2z\,k$$

At (1, 2, 1), $\nabla\phi = 32\,i + 16\,j + 32\,k$

And

$$\frac{a}{|a|} = \frac{2i + j + 2k}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \frac{2i + j + 2k}{\sqrt{9}} = \frac{2i + j + 2k}{3} = \frac{1}{3}(2i + j + 2k).$$

Hence, from (1), we have

$$\frac{d\phi}{ds} = \frac{1}{3}(2i + j + 2k)\cdot(32i + 16j + 32k) = \frac{1}{3}\{(2)(32) + (1)(16) + (2)(32)\} = 48.$$

Example 03: Find a unit vector normal to the surface $\phi(x,y,z) = x^3y^2 - z^2yx^2 - 9 = 0$ at (1, 2, 1).

Solution: We know that if ϕ (x, y, z) = c is a surface then normal vector to this surface is the gradient of □(x, y, z). Now, $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3y^2 - z^2yx^2 - 9)$

➔ $$\nabla\phi = i\frac{\partial}{\partial x}(x^3y^2 - z^2yx^2 - 9) + j\frac{\partial}{\partial y}(x^3y^2 - z^2yx^2 - 9) + k\frac{\partial}{\partial z}(x^3y^2 - z^2yx^2 - 9)$$

$$=\left(3x^2y^2-2xyz^2\right)i+\left(2x^3y-x^2z^2\right)j+\left(-2x^2yz\right)k$$

At (1, 2, 1), $\nabla\phi = 8i + 3j - 4k$

Also, $|\nabla\phi| = \sqrt{8^2+3^2+(-4)^2} = \sqrt{89}$

Hence a unit vector normal to the give surface is

$$u=\frac{(\nabla\phi)_{(1,2,1)}}{\left|(\nabla\phi)_{(1,2,1)}\right|}=\frac{8i+3j-4k}{\sqrt{(8)^2+(3)^2+(-4)^2}}=\frac{8i+3j-4k}{\sqrt{89}}=\frac{1}{\sqrt{89}}(8i+3j-4k).$$

The Divergence and Curl of a Vector Function

In the previous section, we have seen that the differential operator ∇ operates on a scalar function to produce a vector function. In this section we shall see that this operator will convert a vector function into a scalar as well as vector when it applies on it.

Definition: Let $f(x,y,z)=f_1i+f_2j+f_3k$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then divergence of **f** written ∇ . **f** or div **f** is defined by

$$\nabla\cdot\mathbf{f}=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot(f_1i+f_2j+f_3k)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}.$$

Definition: If $\mathbf{f}(x,y,z)=f_1i+f_2j+f_3k$ is a differentiable vector field then the curl or rotation of **f** is denoted by Curl **f** or $\nabla\times\mathbf{f}$ or (rot **f**) and is defined as

$$\text{Curl}\,\mathbf{f}=\nabla\times\mathbf{f}=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\times(f_1i+f_2j+f_3k)$$

$$=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=i\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_2 & f_3\end{vmatrix}-j\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ f_1 & f_3\end{vmatrix}+k\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ f_1 & f_2\end{vmatrix}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)i-\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)j+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)k$$

Note that in the expansion of the determinant, the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ must precede $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$.

Solenoid and Irrotational Vector Functions

Definition: If divergence of a vector function **f** is zero, that is, ∇ . **f** $= 0$ then **f** is called Solenoid Vector Function.

Example 04: For what value of constant P, the vector function defined as F = $2x^2yz\,i + x^2y^3\,j - Pxyz^4\,k$ is solenoid at (1, 2,3).

Solution: We know that if vector function **f** is solenoid then div **f** = 0. Now,

$$\nabla\cdot F=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot\left(2x^2yz\,i+x^2y^3j+xyz^4k\right)=\frac{\partial}{\partial x}\left(2x^2yz\right)+\frac{\partial}{\partial y}\left(x^2y^3\right)+\frac{\partial}{\partial z}\left(xyz^4\right)$$

$$=4xyz+3x^2y^2-4P\,xyz^3$$

Since F is solenoid, therefore $4xyz+3x^2y^2-4P\,xyz^3=0$ At (1, 2, 3)

$$4(1)(2)(3)+3(1)^2(2)^2-4P(1)(2)(3)^3=0\Rightarrow 36-216P=0\Rightarrow P=1/6.$$

Definition: Let f(x, y, z) be a vector field. If Curl **f** is a zero vector, that is, then **f** is called Irrotational vector function.

Example 05: If f = (x + 2y + az) i + (bx – 3y – z) j + (4x + cy + 2z) k is irrotational vector find a,b and c.

Solution: Given **f** = (x + 2y + az) **i** + (bx – 3y – z) **j** + (4x + cy + 2z) **k**. By definition,

$$\text{Curl}\,\mathbf{f} = \nabla\times\mathbf{f} = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\times\left[(x+2y+az)i+(bx-3y-z)j+(4x+cy+2z)k\right]$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$=i\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right]-j\left[\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+k\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]$$
$$=(c+1)i-(4-a)j+(b-2)k=(c+1)i+(a-4)j+(b-2)k$$

If **f** is irrotational, then $\nabla\times\mathbf{f}=0 \quad \Rightarrow c+1=0, a-4=0, b-2=0$ ➔ a = 4, b = 2 and c = -1

Example 06: Show that div r^n r = (n + 3) r^n.

Solution: We know that r = x**i** + y**j** + z**k** ➔ $r=\sqrt{x^2+y^2+z^2}$. Then,

$$\frac{\partial r}{\partial x}=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}.2x=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}.$$ Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

Now, r^n **r** = (x**i** + y**j** + z**k**) r^n = xr^n **i** + yr^n **j** + zr^n **k**. Hence,

$$\text{div}\left(r^n\mathbf{r}\right)=\nabla.\left(r^n\mathbf{r}\right)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot\left(xr^ni+yr^nj+zr^nk\right)=\left(\frac{\partial}{\partial x}xr^n+\frac{\partial}{\partial y}yr^n+\frac{\partial}{\partial z}zr^n\right) \quad (1)$$

Now $\frac{\partial}{\partial x}\left(xr^n\right)=r^n\frac{\partial}{\partial x}x+x\frac{\partial}{\partial x}r^n=r^n.1+xnr^{n-1}\frac{\partial r}{\partial x}=r^n+xnr^{n-1}.\frac{x}{r}=r^n+nx^2r^{n-2}$

Similarly, $\frac{\partial}{\partial y}\left(yr^n\right)=r^n+ny^2r^{n-2}$ and $\frac{\partial}{\partial z}\left(zr^n\right)=r^n+nz^2r^{n-2}$

Thus equation (1) becomes,

div (r^n **r**) = $r^n + nx^2 r^{n-2} + r^n + ny^2 r^{n-2} + r^n + nz^2 r^{n-2} = 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$
$= 3r^n + nr^{n-2} . r^2 = (3 + n) r^n$.

Two Important Theorems

Prove that: 1. $\text{div}(\text{curl}\,\mathbf{f})=0$ **2.** curl $(\text{grad}\,\phi)=0$

Proof: (1) By definition,

$$\text{div}(\text{curl}\,\mathbf{f})=\nabla\cdot(\nabla\times\mathbf{f})=\left[\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right].\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}=0$$

The result is zero as we see that to rows of the determinant are identical.

(2) By definition,

$$\text{curl}\,(\text{grad}\phi)=\nabla\times(\nabla\phi)=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$=\left(\partial^2\phi/\partial y\partial z-\partial^2\phi/\partial z\partial y\right)i-\left(\partial^2\phi/\partial x\partial z-\partial^2\phi/\partial z\partial x\right)j+\left(\partial^2\phi/\partial y\partial x-\partial^2\phi/\partial x\partial y\right)k=(0)i-(0)j+(0)k=0$$

∇^2 - Operator

We know that "Nabla Operator" is a vector operator and is defined as: $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$.

We define $\nabla^2 = \nabla.\nabla = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$

Example 07: Evaluate $\nabla^2(r)$, where $r = \sqrt{x^2+y^2+z^2}$ is the absolute value of position vector $\mathbf{r}$ = **xi + yj + zk.**

Solution: By definition $\nabla^2(r) = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}$. Now,

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}r}{r^2} = \frac{r.1 - x\frac{x}{r}}{r^2} = \frac{r^2-x^2}{r^3} \qquad \text{NOTE: } \frac{\partial}{\partial x}r = \frac{x}{r}$$

Similarly, $\frac{\partial^2 r}{\partial y^2} = \frac{r^2-y^2}{r^3}$ and $\frac{\partial^2 r}{\partial z^2} = \frac{r^2-z^2}{r^3}$. Thus,

$$\nabla^2(r) = \frac{r^2-x^2+r^2-y^2+r^2-z^2}{r^3} = \frac{3r^2-(x^2+y^2+z^2)}{r^3} = \frac{3r^2-r^2}{r^3} = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

WORKSHEET 10

1. Find the vector whose magnitude is that of the vector **5i – 3j + 9k** $5\hat{i}-3\hat{j}+9\hat{k}$ and is in the direction of the vector **4i – 3j + k**.
2. Find the value of *t* if the vectors **5i – 3j + 9k** and **4i – tj + k** have same direction.
3. For what value of p the vectors **a** = [2, 4, -7] and **b** = [2, 6, p] are perpendicular?
4. Find the area of the parallelogram determined by $3\hat{i}+4\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$.
5. Find the volume of parallelepiped if **a** = [3, 4, 0], **b** = [2, 3, 4] and **c** = [0, 0, 5] are its edges.
6. Find the volume of the tetrahedron whose vertices are the points A(2, -1, -3), B(4, 1, 3), C(3, 2, -1) and D(1, 4, 2).
7. The coordinates of a moving particle are given by $x = 4t - t^2/2$, $y = 3 + 6t - t^3$ and $z = 3t^2$. Find the velocity and acceleration of the particle when t = 2 sec.
8. A particle moves so that its displacement at time *t* is given by: x(t) = 2cos t **i** + 2sin t **j** + t**k**
Find the magnitude of the velocity and acceleration of the particle at t = 0.
9. An acceleration of a particle is given by: **f**``(t) = t^2**i** + t^2**j** – (t – 2)**k**
Find the velocity and displacement vector given that **f**`(0) = **i** + **j** + **k** and **f**(0) = 2**i** + 3**j** – **k**.
10. Define gradient, divergence and curl. If ⊓ = $3x^2yz^2$, find $|\nabla\phi|$ at (2, 1, 2).
11. Find a unit vector normal to the surface $x^3y^2 - z^2yx^2 = 8$ at (1, 2, 1).
12. Find a normal vector of magnitude 5 to the surface $x^2y^2 + xy^2z = 10$ at (1, 2, 1).
13. Find the directional derivative of $f = 4x^2y^2z^2$ at (1,2,1) in the direction of 2**i** + **j** + 2**k**.
14. Find the directional derivative of $f = x^2y^2 + y^2z^2$ at (1,1,2) in the direction of **i** + **j** + **k**.
15. Find the value of the constant "*C*" so that the vector function **f** = x^2y^2z **i** + $(x + Cy^2)$ **j** – xyz^2 **k** is Solenoid at (1, 2, 1).
16. For what value of constant "*C*", the vector function **f** = (3x + y) **i** + (Cy + z) **j** + 2z **k** is Solenoid.
17. Show that div **r** = 3 and div(**r**/r^3) = 0 where **r** is a position vector.
18. Show that the curl of the position vector is zero, that is; Curl **r** = 0.
19. Prove that $\nabla r^n = n\, r^{n-2}$ **r**, where **r** is a position vector.
20. Show that Curl r^n **r** = 0.
21. Evaluate: (a) $\nabla^2 r^2$ (b) $\nabla^2 \ln r$.

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